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# BRST approach to gauge invariant higher spin theory of conformal field in flat space

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We propose a closed higher-spin algebra and its representation that reproduces conformal invariant Lagrangian presented by Fradkin and Tseytlin. We use this algebra for constructing gauge invariant Lagrangian by BRST method. Lagrangian constructed by BRST method does not have any off-shell constraints or higher derivative terms as in the non-conformal case. As an example for spin 2 case in four space-time dimension, our Lagrangian agrees with that of conformal gravity by using gauge fixing and equations of motions of auxiliary fields.

**Keywords:** *higher-spin, BRST, conformal.*

## 1 Introduction

Higher-spin field theory is expected to include particles with spin higher than two [1]. In particular, the theory we consider here is a spin independent formalism of particles. It is defined in the same form for spin one vector gauge field, spin two gravitational field and higher spin fields those were not observed. To construct Lagrangian of higher-spin field, we here use, so called, BRST method. Free Lagrangian construction is well developed by using this method in the case that the Poincare symmetry is space-time symmetry, for bosonic case, fermionic case and mixed symmetric case [2] [3].

A spin $s$ conformal higher spin theory in a free Lagrangian form was proposed by Fradkin and Tseytlin [4] in four space-time dimension[1]

$$\mathcal{L} = \phi_{\mu_1\cdots\mu_s}\Box^s P^{\mu_1\cdots\mu_s}_{\nu_1\cdots\nu_s}\phi^{\nu_1\cdots\nu_s}, \tag{1}$$

where $P$ is a projector for traceless and divergence free field. The structure of $P$ considered there was very implicit.

Our motivation is to reproduce this Lagrangian with explicit form. For that purpose BRST method may be a good choice. In this paper we develop BRST method for constructing gauge invariant Lagrangian with conformal symmetry.

In section 2 we start by a set of constraints suggested from (1) and find a closed Higher-spin algebra. In section 3, BRST operator and Lagrangian will be constructed. For example in section 4, the Lagrangian agrees with that of linearized version of conformal gravity in spin two and in four space-time dimension.

[1]It is also given in arbitrary dimension by [5].

## 2 Higher spin algebra

Lagrangian(1) and [4] [5] suggest a set of constraints those the basic field must satisfy. They are d'Alembertian constraint, divergence free constraint and traceless constraint:

$$\begin{array}{lcl} \Box^{s+\frac{D-4}{2}}\phi_{\mu_1\cdots\mu_s} & = & 0 \\ \partial^{\mu_1}\phi_{\mu_1\cdots\mu_s} & = & 0 \\ \eta^{\mu_1\mu_2}\phi_{\mu_1\mu_2\cdots\mu_s} & = & 0 \end{array}. \tag{2}$$

Last two constraints are the same to the constraint for higher-spin field for Poincare symmetry. Only difference from Poincare case is in the first one.

In BRST method we rewrite these constraints on a Fock space by introducing several set of creation-annihilation operators. In order to find a closed algebra(higher-spin algebra) corresponding to (2), we need to look for appropriate representation that defines algebra. Difference of the first equation of (2) from Poincare case leads different higher-spin algebra and its representation.

As a result, we found them in the following. First define Fock space:

$$|\Phi\rangle = \sum_{\{k\}=0}^{\infty} c_+^{\dagger\,k_+} c_-^{\dagger\,k_-} b_1^{\dagger\,k_1} b_2^{\dagger\,k_2} a^\dagger_{\mu_1}\cdots a^\dagger_{\mu_{k_0}}|0\rangle\Phi^{\mu_1\cdots\mu_{k_0}}_{k_-,k_+,k_0,k_1k_2}, \tag{3}$$

where
$[a^\mu, a^{\dagger\nu}] = -\eta^{\mu\nu}$, $[b_I, b_I^\dagger] = [c_-, c_+^\dagger] = [c_+, c_-^\dagger] = 1$, $a^\mu|0\rangle = b_I|0\rangle = c_\pm|0\rangle = 0$, $(I = 1, 2)$.

Then define representation of generators:

$$\begin{array}{lcl} L_0 & = & -p^2 + c_-^\dagger c_- \\ L_1 & = & ap + c_-^\dagger b_1 \\ L_1^+ & = & a^\dagger p + c_- b_1^\dagger \\ L_2 & = & c_- \frac{a^2}{2} + c_-^\dagger \left( -\frac{b_1^2}{2} + (b_2^\dagger b_2 - s - \frac{D-5}{2}) b_2 \right) \\ L_2^+ & = & c_-^\dagger \frac{a^{\dagger 2}}{2} + c_- \left( -\frac{b_1^{\dagger 2}}{2} + b_2^\dagger \right) \\ G_0 & = & -a^\dagger a + b_1^\dagger b_1 + 2 b_2^\dagger b_2 - s + 3 \\ L_c & = & b_1^\dagger b_1 + 2 b_2^\dagger b_2 + c_+^\dagger c_- + c_-^\dagger c_+ - s + 4 - \frac{D}{2} \end{array} . \tag{4}$$

These define the following higher-spin algebra.

$$\begin{array}{lclllcl} [L_0, \mathrm{all}] & = & 0 & & [L_2, L_1^+] & = & -c_- L_1 \\ [L_1, L_1^+] & = & L_0 & & [L_2^+, L_1] & = & c_-^\dagger L_1^+ \\ & & & & [L_2, L_2^+] & = & c_-^\dagger c_- G_0 \\ & & & & & & \\ [G_0, L_1] & = & -L_1 & & [L_c, L_2] & = & -L_2 \\ [G_0, L_1^+] & = & L_1^+ & & [L_c, L_2^+] & = & L_2^+ \\ [G_0, L_2] & = & -2L_2 & & [L_c, L_1] & = & 0 \\ [G_0, L_2^+] & = & 2L_2^+ & & [L_c, G_0] & = & 0 \end{array} . \tag{5}$$

Constraints in Fock space are then written as:

$$L_0|\Phi\rangle = L_1|\Phi\rangle = L_2|\Phi\rangle = G_0|\Phi\rangle = L_c|\Phi\rangle = 0. \tag{6}$$

Constraint by $G_0$ and $L_c$ are explained as fixing spin and conformal weight of state respectively.
This algebra (5) is closed and found to be a natural modification of massive Poincare higher-spin algebra. Structure constant depends on oscillators $c_-$ and $c_-^\dagger$. It is, however, not harmful because the right hand sides of commutators in (5) do not give new constraints. Algebra itself does not depend on spin or conformal weight. Representation, however, depends on them through two arbitrary parameters $s$ and $D$ and one can choose them freely.

## 3 BRST construction of Lagrangian

BRST nilpotent operator is defined in the usual manner [2] [3]:

$$\begin{array}{rl} \tilde{Q} = & \eta_0 L_0 + \eta_1^+ L_1 + \eta_1 L_1^+ + \eta_2^+ L_2 + \eta_2 L_2^+ \\ & +\eta_G G_0 + \eta_c L_c - \eta_1 \eta_1^+ P_0 - c_-^\dagger c_- \eta_2 \eta_2^+ P_G \\ & +(\eta_G \eta_1^+ + c_- \eta_2^+ \eta_1) P_1 + (\eta_1 \eta_G + c_-^\dagger \eta_1^+ \eta_2) P_1^+ \\ & +2\eta_G \eta_2^+ P_2 + 2\eta_2 \eta_G P_2^+ + \eta_c \eta_2^+ P_2 + \eta_2 \eta_c P_2^+ \end{array} , \tag{7}$$

where $\{\eta_i, P_i^\dagger\} = \{\eta_j, P_j\} = 1$, $\eta_I|0\rangle = P_I|0\rangle = P_j|0\rangle = 0$, $(I = 1, 2;\ j = 0, G, c)$.
BRST equation and gauge transformation are written in the extended Fock space that is defined by multiplying anti-commuting ghost operators $\eta$'s, $P$'s on $|\Phi_{0\cdots 0}\rangle \equiv |\Phi\rangle$ of (3):

$$\tilde{Q}|\chi^0\rangle = 0\ , \quad \delta|\chi^0\rangle = \tilde{Q}|\chi^1\rangle, \delta|\chi^1\rangle = \tilde{Q}|\chi^2\rangle, \cdots \tag{8}$$

$$|\chi^I\rangle = \sum_{\substack{k_3 + k_4 + k_6 \\ -k_5 - k_7}=-I} \eta_0^{k_3} \eta_1^{\dagger k_4} P_1^{\dagger k_5} \eta_2^{\dagger k_6} P_2^{\dagger k_7} |\Phi_{k_3 \cdots k_7}\rangle, \tag{9}$$

where ghost number restriction is expressed in above summation. To find Lagrangian we need a treatment of Hermitian ghosts as in the bosonic Poincare case. Our case has an additional pair of Hermitian ghosts. We treat them as we did for Poincare case. Then BRST equation (8) is divided to three parts:

$$Q|\chi^I\rangle = \sigma_g|\chi^I\rangle = \sigma_c|\chi^I\rangle = 0, \tag{10}$$

where $\sigma_g$, $\sigma_c$ and $Q$ are defined by using number operators of corresponding creation-annihilation operators:

$$\begin{array}{l} \tilde{Q} \equiv Q + \eta_G \sigma_s + \eta_c \sigma_c - c_-^\dagger c_- \eta_2^\dagger \eta_2 P_G, Q^2|\chi^i\rangle = 0 \\ \sigma_g + s \equiv n_a + n_{b_1} + 2n_{b_2} + n_{\eta_1} + n_{P_1} + 2n_{\eta_2} + 2n_{P_2}. \\ \sigma_c + s + \frac{D-6}{2} \equiv n_{b_1} + 2n_{b_2} + n_{c_+} + n_{c_-} + n_{\eta_2} + n_{P_2} \end{array} \tag{11}$$

$\sigma_g$ and $\sigma_c$ fix spin and space-time dimension. Lagrangian and gauge transformation for fixed spin and space-time dimension are then written, by introducing some operator $K$ for Hermiticity [2], as

$$\mathcal{L} = \int d\eta_0 \langle\chi^0|KQ|\chi^0\rangle \tag{12}$$

$$\delta|\chi^0\rangle = Q|\chi^1\rangle, \delta|\chi^1\rangle = Q|\chi^2\rangle. \tag{13}$$

These have the same form to Poincare case. All differences are included in the structure of BRST operator and states.

## 4 Example: spin two in four space-time dimension

In this section we write down Lagrangian (12) for spin two in four space-time dimension and show it reproduces Lagrangian of linearized conformal gravity in flat space-time.
The second and the third equation of (10) as well as the ghost number restriction are, by setting $s = 2$ and $D = 4$, solved for states as

$$\begin{array}{lcl} |\chi^0\rangle & = & |S^0\rangle_{2,1} + \eta_0|A^0\rangle_{2,1} \\ |\chi^1\rangle & = & |S^1\rangle_{2,1} + \eta_0|A^1\rangle_{2,1} \\ |\chi^2\rangle & = & 0 \end{array} , \tag{14}$$

where lower indexes of states represent eigenvalues of right hand sides of last two equations of (11). We expand these states by ghost's dependencies as:

$$\begin{array}{rcl} |S^0\rangle_{2,1} &=& |S_1\rangle_{2,1} + \eta_1^\dagger P_1^\dagger |S_2\rangle_{0,1} \\ |A^0\rangle_{2,1} &=& P_1^\dagger |A_1\rangle_{1,1} + P_2^\dagger |A_2\rangle_{0,0} \\ |S^1\rangle_{2,1} &=& P_1^\dagger |S_1^1\rangle_{1,1} + P_2^\dagger |S_2^1\rangle_{0,0} \end{array} . \qquad (15)$$

One can eliminate $|A_1\rangle_{1,1}$ by its equation of Motion:

$$|A_1\rangle_{1,1} \;=\; L_1|S_1\rangle_{2,1} - L_1^+|S_2\rangle_{0,1} . \qquad (16)$$

Then Lagrangian and gauge transformations are

$$\begin{array}{rl} \mathcal{L} = & \langle S_1|K(L_0 - L_1^+ L_1)|S_1\rangle - 2\langle S_2|KL_0|S_2\rangle \\ + & \left(\langle S_1|KL_1^{+2}|S_2\rangle - \langle A_2|K\left(L_2|S_1\rangle + c_-|S_2\rangle\right) + c.c.\right) \end{array} \qquad (17)$$

$$\begin{array}{rcl} \delta|S_1\rangle &=& L_1^+|S_1^1\rangle + L_2^+|S_2^1\rangle \\ \delta|S_2\rangle &=& L_1|S_1^1\rangle - c_-^+|S_2^1\rangle \\ \delta|A_2\rangle &=& L_0|S_2^1\rangle \end{array} . \qquad (18)$$

We further expand them by $c_\pm^\dagger$ and $b_1^\dagger$:

$$\begin{array}{rcl} |S_1\rangle_{2,1} &=& c_-^\dagger|S_1^-\rangle_{2,0} + c_+^\dagger|S_1^+\rangle_{2,0} + b_1^\dagger|S_1^0\rangle_{1,0} \\ |S_2\rangle_{0,1} &=& c_-^\dagger|S_2^-\rangle_{0,0} + c_+^\dagger|S_2^+\rangle_{0,0} \\ |S_1^1\rangle_{1,1} &=& c_-^\dagger|S_1^{1,-}\rangle_{1,0} + c_+^\dagger|S_1^{1,+}\rangle_{1,0} + b_1^\dagger|S_1^{1,0}\rangle_{0,0} \end{array} . \qquad (19)$$

Next we use equation of Motion of $|A_2\rangle_{0,0}$, then using degrees of freedom of$|S_1^{1,+}\rangle_{1,0}$ and $|S_1^{1,0}\rangle_{0,0}$ we eliminate $|S_1^0\rangle_{1,0}$ and $|S_2^+\rangle_{0,0}$ respectively. By using equation of motion of $\langle S_1^+|$, higher derivative term appears:

$$\begin{array}{rcl} \mathcal{L} &=& \langle S_1^-|\left(l_0 - l_1^+ l_1 - l_1^{+2} l_2\right) \\ &\times& \left(1 - \frac{2}{3} l_2^+ l_2\right)\left(l_0 - l_1^+ l_1 - l_2^+ l_1^2\right)|S_1^-\rangle , \\ \delta|S_1^-\rangle &=& l_1^+|S_1^{1,-}\rangle + l_2^+|S_2^1\rangle, \end{array} \qquad (20)$$

where lower case $l$'s are the generators of massless Poincare one. We can write usual fields instead of states and operators:

$$\begin{array}{l} |S_1^-\rangle = \frac{(-i)^2}{2!} a^{+\mu} a^{+\nu}|0\rangle h_{\mu\nu}(x) \\ |S_1^{1,-}\rangle = -ia^{+\mu}|0\rangle\lambda_\mu(x), |S_2^1\rangle = |0\rangle\delta(x) \end{array} . \qquad (21)$$

Then Lagrangian is written by familiar form:

$$2\mathcal{L} = h^{\mu\nu}\partial^4 h_{\mu\nu} - 2(h\partial)_\mu \partial^2(\partial h)^\mu + \frac{2}{3} h_\lambda^\lambda \partial^2(\partial\partial h) + \frac{2}{3}(h\partial\partial)(\partial\partial h) - \frac{1}{3} h_\lambda^\lambda \partial^4 h_\lambda^\lambda . \qquad (22)$$

This agrees with the Lagrangian of linearised conformal gravity in flat space-time and it can be written as (1) with the projector.

## 5 Conclusion

We proposed a closed higher-spin algebra and its representation that reproduces the conformal invariant higher-spin Lagrangian presented by Fradkin and Tseytlin [4].
We started a set of constraints (2) suggested by (1) and found a closed higher-spin algebra (5).
Some properties of the algebra (5) are the following. It looks like a natural modification of massive Poincare higher-spin algebra. It does not depend on spin or conformal weight. It is defined for arbitrary spin in any space-time dimension. It may correspond to a sub algebra of full conformal algebra because, compared to Poincare case, we have add only one additional constraint by $L_c$ that may correspond dilatation. Since we did not consider one for special conformal transformation, full conformal Higher-spin algebra will possibly be modified from (5). For spin one or two case, however, differences will not be important. In fact, for spin two, our higher-spin algebra is enough to reproduce known Lagrangian of conformal gravity.
We have used this algebra for constructing free gauge invariant Lagrangian by BRST method. The Lagrangian (12) constructed by BRST method does not have any off-shell constraints or higher derivative terms as in the all Poincare cases. The structure of states in our Lagrangian is similar to that of Metsaev [6]. The number of terms in our states, however, are initially not limited by definition (3). It is determined by BRST equation of motion from Lagrangian.
In order to check the validity of higher-spin algebra we proposed, we studied an example of spin two in four space-time dimension. We wrote down generally defined Lagrangian for this case. By using gauge fixing and equations of motions of auxiliary fields, it was reproduced that the known form of Lagrangian of linearized conformal gravity (22) or of Fradkin and Tseytlin's. It is also possible to study spin one case in arbitrary space-time dimension and the known form can be given too. We expect that the algebra (5) also reproduces Lagrangian (1) and that of [5] for general integer spin in arbitrary space-time dimension.
One of interesting future work is to clarify the relation between the full conformal algebra and higher-spin algebra.

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## БРСТ ПОДХОД К КАЛИБРОВОЧНО-ИНВАРИАНТНЫМ ТЕОРИЯМ КОНФОРМНЫХ ПОЛЕЙ ВЫСШИХ СПИНОВ В ПЛОСКОМ ПРОСТРАНСТВЕ

Предлагается замкнутая алгебра высших спинов и ее представление, которое воспроизводит конформно инвариантный лагранжиан, полученный Фрадкиным и Цейтлиным. Эта алгебра используется для построения калибровочно инвариантного лагранжиана с помощью БРСТ подхода. Лагранжиан, построенный с помощью метода БРСТ-конструкции не имеет ни связей вне массовой оболочки ни членов с высшими производным по сравнению с неконформным случаем. На примере спина 2 в четырех измерениях наш лагранжиан согласуется с тем, что следует из конформной гравитации с использованием калибровки и уравнений движений на вспомогательные поля.

**Ключевые слова:** *высшие спины, БРСТ, конформная симметрия.*

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