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NSVZ SCHEME AND THE REGULARIZATION BY HIGHER DERIVATIVES

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The NSVZ scheme is constructed in all orders for the renormalization group functions defined in terms of the renormalized coupling constant for Abelian $\mathcal{N} = 1$ supersymmetric theories regularized by higher derivatives. For the other renormalization prescriptions the scheme-independent consequences of the NSVZ relation are investigated. It is explained, why for the renormalization group functions defined in terms of the bare coupling constant the NSVZ relation is valid for all renormalization prescriptions in the case of using the higher derivative regularization.

Keywords: *supersymmetry, renormalization, β-function, anomalous dimension.*

1 Introduction

The NSVZ β-function [1–4] is a relation between the β-function of $\mathcal{N} = 1$ supersymmetric theories and the anomalous dimensions of the matter superfields:

$$\beta(\alpha) = -\frac{\alpha^2\Big(3C_2 - T(R) + C(R)_i{}^j\gamma_j{}^i(\alpha)/r\Big)}{2\pi(1 - C_2\alpha/2\pi)}. \qquad (1)$$

Here we use the notation

$$\mathrm{tr}\,(T^AT^B) \equiv T(R)\,\delta^{AB}; \quad (T^A)_i{}^k(T^A)_k{}^j \equiv C(R)_i{}^j; \quad f^{ACD}f^{BCD} \equiv C_2\delta^{AB}; \qquad r \equiv \delta_{AA}. \qquad (2)$$

For the particular case of $\mathcal{N} = 1$ supersymmetric electrodynamics (SQED) with N_f flavors the NSVZ β-function takes the form [5, 6]

$$\beta(\alpha) = \frac{\alpha^2 N_f}{\pi}\Big(1 - \gamma(\alpha)\Big). \qquad (3)$$

The NSVZ β-function was constructed using various general arguments: structure of instanton contributions [1, 3, 7], anomalies [2, 4, 8], the non-renormalization theorem for the topological term [9].

The NSVZ expression can be compared with the results of explicit calculations which in supersymmetric theories are mostly made using the regularization by the dimensional reduction [10]. (It should be noted that this regularization is either mathematically inconsistent [11], or is not manifestly supersymmetric [12] and can break supersymmetry in higher loops [13, 14].) Using the dimensional reduction supplemented by the $\overline{\mathrm{DR}}$-scheme the β-function for general $\mathcal{N} = 1$ supersymmetric theories was calculated up to the four-loop approximation [15–18]. The NSVZ β-function agrees with these calculations only in the one- and two-loop approximations. In the higher loops it is obtained only after a specially tuned finite renormalization [16, 19].

It appears that a very convenient tool for calculating quantum corrections in supersymmetric theories is the higher covariant derivative regularization [20, 21]. (It also includes the Pauli-Villars regularization for removing the one-loop divergences [22, 23].) Unlike the dimensional reduction, it is consistent and (if it is used for supersymmetric theories) does not break supersymmetry [24, 25]. This regularization can be also formulated for $\mathcal{N} = 2$ supersymmetric theories [26, 27].

The explicit calculations made with the higher derivative regularization in $\mathcal{N} = 1$ supersymmetric theories reveal an interesting feature of quantum corrections: integrals giving the β-function defined in terms of the bare coupling constant are integrals of (double) total derivatives [28, 29]. (Note that in these integrals the external momentum vanishes.) The NSVZ relation appears after calculating the momentum integral of a total derivative. For Abelian supersymmetric theories this was proved exactly in all orders [30, 31].

However, the renormalization group (RG) functions defined by the standard way in terms of the renormalized coupling constant [32] are scheme dependent. They satisfy the NSVZ relation only with a certain subtraction scheme, which is called the NSVZ scheme.

At present there is no general prescription how to construct this scheme in all orders with the dimensional reduction. However this can be easily done using the higher derivative regularization [33]. In the present paper we describe how this can be made.

2 $\mathcal{N}=1$ SQED with N_f flavors, regularized by higher derivatives

In this paper we consider $\mathcal{N}=1$ SQED with N_f flavors which is described by the action

$$S=\frac{1}{4e_0^2}\mbox{Re}\int d^4x\,d^2\theta\,W^aW_a+\sum_{i=\alpha}^{N_f}\frac{1}{4}\int d^4x\,d^4\theta \times\Big(\phi_\alpha^* e^{2V}\phi_\alpha+\widetilde\phi_\alpha^* e^{-2V}\widetilde\phi_\alpha\Big), \qquad (4)$$

in the massless limit. Here V is a real gauge superfield, ϕ_α and $\widetilde\phi_\alpha$ with $\alpha=1,\ldots,N_f$ are chiral matter superfields. In the Abelian case $W_a=\bar D^2 D_a V/4$. In order to introduce the higher derivative regularization we add the higher derivative term S_Λ to the classical action:

$$S_{\mbox{\scriptsize reg}}=S+S_\Lambda, \qquad (5)$$

where

$$S_\Lambda=\frac{1}{4e_0^2}\mbox{Re}\int d^4x\,d^2\theta\,W^a\left(R(\partial^2/\Lambda^2)-1\right)W_a \qquad (6)$$

and the function $R-1$ contains the large degree of derivatives. A convenient choice of this function is $R=1+\partial^{2n}/\Lambda^{2n}$.

By introducing S_Λ one regularizes all divergences beyond the one-loop approximation. The remaining one-loop divergencies can be removed by inserting the Pauli-Villars determinants into the generating functional [23]:

$$Z[J,\mathbf{\Omega}]=\int D\mu\prod_I\Big(\det PV(V,M_I)\Big)^{N_f c_I} \times\exp\Big\{iS_{\mbox{\scriptsize reg}}+iS_{\mbox{\scriptsize gf}}+S_{\mbox{\scriptsize Sources}}\Big\}. \qquad (7)$$

We require that the degrees of the Pauli-Villars determinants c_I satisfy the constrains

$$\sum_I c_I=1;\qquad \sum_I c_I M_I^2=0 \qquad (8)$$

due to which the remaining one-loop divergences cancel. The masses of the Pauli-Villars fields are chosen proportional to the parameter Λ, the ratios being independent of the bare coupling constant:

$$M_I=a_I\Lambda,\qquad a_I\ne a_I(e_0). \qquad (9)$$

Let us define the functions $d^{-1}(\alpha_0,\Lambda/p)$ and $G(\alpha_0,\Lambda/p)$ according to the following equation:

$$\Gamma^{(2)}=\int\frac{d^4p}{(2\pi)^4}\,d^4\theta\,\Big(-\frac{1}{16\pi}V(-p,\theta)\,\partial^2\Pi_{1/2} \times V(p,\theta)d^{-1}(\alpha_0,\Lambda/p)+\frac{1}{4}\sum_{\alpha=1}^{N_f}\Big(\phi_\alpha^*(-p,\theta) \times\phi_\alpha(p,\theta)+\widetilde\phi_\alpha^*(-p,\theta)\widetilde\phi_\alpha(p,\theta)\Big)\,G(\alpha_0,\Lambda/p)\Big), \qquad (10)$$

where $\Gamma^{(2)}$ is a part of the effective action corresponding to the two-point Green functions and $\partial^2\Pi_{1/2}$ denotes a supersymmetric transversal projection operator.

In order to construct the renormalized coupling constant $\alpha(\alpha_0,\Lambda/\mu)$ we require finiteness of $d^{-1}(\alpha_0(\alpha,\Lambda/\mu),\Lambda/p)$ in the limit $\Lambda\to\infty$. The renormalization constant Z_3 is then defined by

$$Z_3(\alpha,\Lambda/\mu)\equiv\frac{\alpha}{\alpha_0}. \qquad (11)$$

Similarly, the renormalization constant Z is constructed by requiring finiteness of the renormalized two-point Green function ZG in the limit $\Lambda\to\infty$.

3 NSVZ relation for the RG functions defined in terms of the bare coupling constant

The RG functions can be defined in terms of the bare coupling constant according to the following prescription:

$$\beta\Big(\alpha_0(\alpha,\Lambda/\mu)\Big)\equiv\frac{d\alpha_0}{d\ln\Lambda}\Big|_{\alpha=\mbox{\scriptsize const}}; \qquad (12)$$

$$\gamma_i{}^j\Big(\alpha_0(\alpha,\Lambda/\mu)\Big)\equiv-\frac{d\ln Z_i{}^j}{d\ln\Lambda}\Big|_{\alpha=\mbox{\scriptsize const}}, \qquad (13)$$

where the derivatives should be calculated at a fixed value of the renormalized coupling constant. It is possible to prove [33] that these RG functions are scheme independent for a fixed regularization, but depend on the regularization. Moreover, in all loops they satisfy the NSVZ relation for Abelian supersymmetric theories, regularized by higher derivatives [30, 31].

The NSVZ relation appears, because with the higher covariant derivative regularization loop integrals giving the β-function defined in terms of the bare coupling constant are integrals of total derivatives [28] and even integrals of double total derivatives [29]. (In these integrals the external momentum vanishes, $p=0$.) As a consequence, one of the momentum integrals can be calculated analytically, producing the NSVZ relation for the RG functions defined in terms of the bare coupling constant:

$$\frac{\beta(\alpha_0)}{\alpha_0^2} = \frac{d}{d\ln\Lambda}\Big(d^{-1}(\alpha_0,\Lambda/p) - \alpha_0^{-1}\Big)\Big|_{p=0}$$
$$= \frac{N_f}{\pi}\Big(1 - \frac{d}{d\ln\Lambda}\ln G(\alpha_0,\Lambda/q)\Big|_{q=0}\Big)$$
$$= \frac{N_f}{\pi}\Big(1-\gamma(\alpha_0)\Big). \qquad (14)$$

Similar features are also valid in the non-Abelian case, but the calculations have been done only in the two-loop approximation [34–36].

4 The NSVZ scheme with the higher derivatives

In the previous section we consider the RG function defined in terms of the bare coupling constant. However, by standard way the RG functions are defined in terms of the renormalized coupling constant [32]:

$$\widetilde{\beta}\Big(\alpha(\alpha_0,\Lambda/\mu)\Big) \equiv \frac{d\alpha}{d\ln\mu}\Big|_{\alpha_0=\text{const}}; \qquad (15)$$
$$\widetilde{\gamma}_i{}^j\Big(\alpha(\alpha_0,\Lambda/\mu)\Big) \equiv \frac{d\ln Z_i{}^j}{d\ln\mu}\Big|_{\alpha_0=\text{const}}. \qquad (16)$$

(In order to obtain these functions it is necessary to express the RHS via α_0 and calculate the derivatives at a fixed value of the bare coupling constant.) The RG functions (15) and (16) are scheme-dependent. According to [33, 37] they coincide with the RG functions defined in terms of the bare coupling constant, if the boundary conditions

$$Z_3(\alpha,x_0)=1; \qquad Z_i{}^j(\alpha,x_0)=1 \qquad (17)$$

are imposed on the renormalization constants, where x_0 is an arbitrary fixed value of $\ln\Lambda/\mu$:

$$\widetilde{\beta}(\alpha)=\beta(\alpha) \qquad \widetilde{\gamma}(\alpha)=\gamma(\alpha). \qquad (18)$$

Due to the scheme-dependence the RG functions $\widetilde{\beta}(\alpha)$ and $\widetilde{\gamma}(\alpha)$ satisfy the NSVZ relation only in a certain subtraction scheme, called the NSVZ scheme. This scheme is evidently fixed in all loops by the boundary conditions (17) if the theory is regularized by higher derivatives, because the functions β and γ satisfy the NSVZ relation in the case of using this regularization.

5 RG functions in the three-loop approximation

Using the higher derivative regularization with $R_k = 1 + k^{2n}/\Lambda^{2n}$ one can calculate the β-function and the anomalous dimension in the three- and two-loop approximations, respectively. Let us present the results for various definitions of the RG functions and in various subtraction schemes.

The RG functions defined in terms of the bare coupling constant coincide with the RG functions defined in terms of the renormalized coupling constant in the NSVZ scheme and are given by the following expressions:

$$\widetilde{\gamma}_{\text{NSVZ}}(\alpha) = \gamma(\alpha) - \frac{\alpha}{\pi} + \frac{\alpha^2}{\pi^2}\Big(\frac{1}{2} + N_f\sum_{I=1}^{n} c_I\ln a_I$$
$$+N_f\Big) + O(\alpha^3); \qquad (19)$$
$$\widetilde{\beta}_{\text{NSVZ}}(\alpha) = \beta(\alpha)\frac{\alpha^2 N_f}{\pi}\Big(1+\frac{\alpha}{\pi} - \frac{\alpha^2}{\pi^2}\Big(\frac{1}{2}+N_f$$
$$\times\sum_{I=1}^{n} c_I\ln a_I + N_f\Big) + O(\alpha^3)\Big). \qquad (20)$$

We see that in this scheme the NSVZ relation is really satisfied in the considered approximation.

Let us also present the results for the RG functions defined in terms of the renormalized coupling constants for other subtraction schemes.

In the MOM scheme the dimensional reduction and the higher derivative regularizations give the same result [37]

$$\widetilde{\gamma}_{\text{MOM}}(\alpha) = -\frac{\alpha}{\pi} + \frac{\alpha^2(1+N_f)}{2\pi^2} + O(\alpha^3); \qquad (21)$$
$$\widetilde{\beta}_{\text{MOM}}(\alpha) = \frac{\alpha^2 N_f}{\pi}\Big(1+\frac{\alpha}{\pi} - \frac{\alpha^2}{2\pi^2}\Big(1+3N_f$$
$$\times(1-\zeta(3))\Big) + O(\alpha^3)\Big). \qquad (22)$$

In the $\overline{\text{DR}}$-scheme the result was obtained in Ref. [16] and is written as

$$\widetilde{\gamma}_{\overline{\text{DR}}}(\alpha) = -\frac{\alpha}{\pi} + \frac{\alpha^2(2+2N_f)}{4\pi^2} + O(\alpha^3); \qquad (23)$$
$$\widetilde{\beta}_{\overline{\text{DR}}}(\alpha) = \frac{\alpha^2 N_f}{\pi}\Big(1+\frac{\alpha}{\pi} - \frac{\alpha^2(2+3N_f)}{4\pi^2} + O(\alpha^3)\Big).$$

Comparing all above expressions one can see that in the considered approximations only terms proportional to $(N_f)^2\alpha^4$ in the β-function and to $N_f\alpha^2$ in the anomalous dimension are scheme dependent. The other terms coincide in all schemes.

6 The NSVZ relation and finite renormalizations

Different renormalization prescriptions can be related by finite renormalizations

$$\alpha\to\alpha'(\alpha); \qquad Z'(\alpha',\Lambda/\mu) = z(\alpha)Z(\alpha,\Lambda/\mu), \qquad (24)$$

under which the β-function (15) and the anomalous dimension (16) are changed as follows:

$$\widetilde{\beta}'(\alpha') = \frac{d\alpha'}{d\alpha}\widetilde{\beta}(\alpha); \qquad \widetilde{\gamma}'(\alpha') = \frac{d\ln z}{d\alpha}\cdot\widetilde{\beta}(\alpha)+\widetilde{\gamma}(\alpha). \quad (25)$$

Using these equations one can see [37] that if $\widetilde{\beta}(\alpha)$ and $\widetilde{\gamma}(\alpha)$ satisfy the NSVZ relation, then

$$\widetilde{\beta}'(\alpha') = \frac{d\alpha'}{d\alpha}\cdot\frac{\alpha^2 N_f}{\pi}\frac{1-\widetilde{\gamma}'(\alpha')}{1-\alpha^2 N_f(d\ln z/d\alpha)/\pi}\Big|_{\alpha=\alpha(\alpha')}.$$

Taking into account that quantum corrections to the coupling constant are proportional at least to N_f we obtain

$$\alpha'(\alpha)-\alpha = O(N_f); \qquad z(\alpha) = O\left((N_f)^0\right). \qquad (26)$$

Therefore, all scheme-dependent terms in the β-function are proportional at least to $(N_f)^2$ in all loops, while the terms proportional to $(N_f)^0$ in the anomalous dimension are scheme-independent. This means that the NSVZ relation is valid for terms proportional to $(N_f)^1$ in all orders. Nevertheless, terms proportional to $(N_f)^\alpha$ with $\alpha \geq 2$ are scheme-dependent.

Similarly making finite renormalizations in the non-Abelian case one can see [38] that in L loops terms proportional to $\mathrm{tr}\left(C(R)^L\right)$ satisfy the NSVZ relation for an arbitrary renormalization prescription. This implies that the NSVZ relation non-trivially constrains the divergences in spite of its scheme-dependence.

7 Conclusion

For Abelian supersymmetric theories, regularized by higher derivatives, the NSVZ β-function relates the scheme-independent RG functions defined in terms of the bare coupling constant. The NSVZ relation follows from the fact that the integrals which determine the β-function defined in terms of the bare coupling constant can be written as integrals of double total derivatives.

For the RG functions defined in terms of the renormalized coupling constant, the NSVZ relation is valid only in a special NSVZ scheme. In the Abelian case the higher derivative regularization enables to give a simple prescription for constructing this scheme in all orders by imposing the boundary conditions (17).

Although the NSVZ relation is scheme-dependent, it has some scheme-independent consequences which non-trivially restrict the divergences.

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NSVZ СХЕМА И РЕГУЛЯРИЗАЦИЯ ВЫСШИМИ ПРОИЗВОДНЫМИ

Для абелевых $\mathcal{N} = 1$ суперсимметричных теорий, регуляризованных высшими производными, NSVZ схема построена во всех порядках для ренормгрупповых функций, определенных в терминах перенормированной константы связи. Для других перенормировочных предписаний исследуются схемно-независимые следствия NSVZ соотношения. Объясняется, почему для ренормгрупповых функций, определенных в терминах голой константы связи, NSVZ соотношение справедливо при произвольных перенормировочных предписаниях в случае использования регуляризации высшими производными.

Ключевые слова: *суперсимметрия, перенормировка, β-функция, аномальная размерность.*

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