UDC 530.1; 539.1

Derivation of the exact NSVZ β-function using effective diagrams

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Using Scwinger–Dyson equation we prove that the β-function for $N = 1$ supersymmetric electrodynamics is given by integrals of double total derivatives. This allows to calculate one of the loop integrals and obtain the exact NSVZ β-function, which relates the β-function and the anomalous dimension of the matter superfield.

Keywords: *supersymmetry, supergraphs, NSVZ β-function.*

1 Introduction

It is well-known that in $N = 1$ supersymmetric theories a β-function is related with anomalous dimensions of the matter superfields [1]. This relation is called the exact Novikov, Shifman, Vainshtein, and Zakharov (NSVZ) β-function. It was originally obtain using arguments based on the structure of instanton contributions or anomalies. In the lowest orders of the perturbation theory this β-function was verified by explicit calculations [2–5]. Most calculations were made using the dimensional reduction in the $\overline{MS}$-scheme. The exact NSVZ β-function agrees with the explicit calculations only in the one- and two-loop approximations. In higher orders for obtaining the exact NSVZ β-function, it is necessary to perform a special redefinition of the coupling constant [6]. This means that there is a so-called NSVZ scheme in which this β-function is obtained. For $N = 1$ supersymmetric electrodynamics (SQED) such a scheme can be naturally constructed using the higher derivative regularization [7]. The higher covariant derivative regularization (unlike the dimensional reduction [8]) is mathematically consistent. It can be generalized to the supersymmetric case [9, 10] so that the supersymmetry is an explicit symmetry of the regularized theory. With the higher derivative regularization integrals defining a β-function can be calculated by differentiating a two-point Green function of the gauge superfield in the limit of vanishing external momentum. It was noted that this integrals are integrals of total derivatives [11] and even integrals of double total derivatives [12]. This features were verified for the general renormalizable $N = 1$ SYM theory in the two-loop approximation by explicit calculations of supergraphs [13,14]. The factorization of integrands into total derivatives allows to calculate one of the loop integrals analytically and relate a β-function with an anomalous dimension of the matter superfields $\gamma(\alpha)$. The result can be obtained exactly in all orders of the perturbation theory [15] and coincides with the exact NSVZ β-function, which for $N = 1$ SQED [16] has the form

$$\beta(\alpha) = \frac{\alpha^2}{\pi}\Big(1 - \gamma(\alpha)\Big). \tag{1}$$

However, in order to generalize the results to non-Abelian case it is more convenient to use a different method based on the Schwinger–Dyson equations [17]. In this paper we use this method in order to prove that a β-function is given by integrals of double total derivatives and coincides with the exact NSVZ β-function.

2 $N = 1$ SQED, regularized by higher derivatives

$N = 1$ SQED is described by the action

$$S = \frac{1}{4e^2}\mathrm{Re}\int d^4x\, d^2\theta\, W^a W_a + \frac{1}{4}\int d^4x\, d^4\theta\,\Big(\phi^* e^{2V}\phi + \widetilde\phi e^{-2V}\widetilde\phi\Big). \tag{2}$$

In order to regularize the theory by higher derivatives, we add to the action a term with the higher derivatives. Then the kinetic term for the gauge superfield will have the form

$$\frac{1}{4e^2}\mathrm{Re}\int d^4x\, d^2\theta\, W^a R(\partial^2/\Lambda^2) W_a, \tag{3}$$

where $R(0) = 1$ and $R(\infty) = \infty$, for example, $R = 1 + \partial^{2n}/\Lambda^{2n}$. After introducing the higher derivative term divergences remain only in the one-loop approximation. In order to cancel them, it is necessary to insert the Pauli–Villars determinants into the generating functional [18]:

$$Z = \int DV\, D\phi\, D\widetilde\phi\, \prod_I (\det(V, M_I))^{c_I} \times \exp\Big(iS_{\text{reg}} + iS_{\text{source}}\Big), \tag{4}$$

where $S_{\rm reg}$ is the regularized action and $S_{\rm source}$ denotes terms with sources. The masses of the Pauli–Villars fields Φ_I are proportional to the parameter Λ, so that there is the only dimensionful parameter in the regularized theory: $M_I = a_I\Lambda$, where a_I are numerical constants, which do not depend on the coupling constant. The coefficients c_I satisfy the conditions

$$\sum_I c_I = 1; \qquad \sum_I c_I M_I^2 = 0. \tag{5}$$

It is convenient to introduce the auxiliary sources ϕ_0 and $\tilde\phi_0$, modifying the action for the matter superfields:

$$\frac{1}{4}\int d^8x\,\Big((\phi^*+\phi_0^*)e^{2V}(\phi+\phi_0)+(\widetilde\phi^*+\widetilde\phi_0^*)e^{-2V}(\widetilde\phi+\widetilde\phi_0)\Big), \tag{6}$$

where the fields ϕ_0 and $\tilde\phi_0$ are not chiral.

The effective action Γ is defined by the standard way. We will also use the Routhian

$$\gamma(J,\phi_i) = W - j^i\cdot\phi_i, \tag{7}$$

where J is a source for the gauge superfield.

With the higher derivative regularization a β-function can be calculated by differentiating the two-point Green function for the gauge superfield with respect to $\ln\Lambda$ in the limit of vanishing external momentum:

$$\frac{d}{d\ln\Lambda}\Big(d^{-1}(\alpha_0,\Lambda/p) - \alpha_0^{-1}\Big)\Big|_{p=0} = -\frac{d\alpha_0^{-1}}{d\ln\Lambda} = \frac{\beta(\alpha_0)}{\alpha_0^2}, \tag{8}$$

where Λ and α are considered as independent variables. This expression is well defined if the RHS is expressed in terms of the bare coupling constant α_0. In order to extract the function d^{-1} it is possible to make the substitution

$$V \to \bar\theta^a\bar\theta_a\theta^b\theta_b \equiv \theta^4. \tag{9}$$

The expression in the RHS does not depend on the space-time coordinates x^μ. Therefore, this substitution automatically lead to the condition $p=0$:

$$(2\pi)^3\delta^4(p)\frac{d}{d\ln\Lambda}\Big(d^{-1}(\alpha_0,\Lambda/p) - \alpha_0^{-1}\Big)\Big|_{p=0} = \frac{d}{d\ln\Lambda}\Big(\Gamma^{(2)}_V - S - S_{\rm gf}\Big)\Big|_{V(x,\theta)=\theta^4}. \tag{10}$$

3 Schwinger–Dyson equations for $N = 1$ SQED

The Schwinger–Dyson equations obtained in [17] can be written as

$$\frac{\delta(\Delta\Gamma)}{\delta V_x} = \Big\langle\frac{\delta S_I}{\delta V_x}\Big\rangle = \frac{1}{2}\Big\langle(\phi^*+\phi_0^*)e^{2V}(\phi+\phi_0) -(\widetilde\phi^*+\widetilde\phi_0^*)e^{-2V}(\widetilde\phi+\widetilde\phi_0)\Big\rangle + (PV), \tag{11}$$

where (PV) denotes contribution of the Pauli–Villars fields and

$$\Delta\Gamma \equiv \Gamma - \frac{1}{4e_0^2}{\rm Re}\int d^4x\,d^2\theta\,W^aW_a - S_{\rm gf}. \tag{12}$$

Differentiating equation (11) with respect to V_y we obtain the Schwinger–Dyson equation for the two-point Green function of the gauge superfield. In a graphical form the result is presented in Fig. 1. Vertexes in these diagrams contain derivatives with respect to auxiliary sources ϕ_0 and $\tilde\phi_0$. In [17] these diagrams were calculated substituting solutions of the Ward identities. This method allows to extract terms which give the exact NSVZ β-function. Explicit calculations (see e.g. [19]) show that the other terms vanish. However, in order to prove this, it is necessary to use some new ideas. Let us briefly describe them here.

1. First, it is necessary to use the Schwinger–Dyson equation one more time for vertexes which contain the derivative $\delta\Gamma/\delta V_y$. This is made using equation (11). Note that in order to simplify the result, it is convenient to write it in terms of the Routhian γ, because in this case the number of effective diagrams is less. A simple graphical interpretation of this procedure is presented in Fig. 2. (Here we do not present expressions for the effective lines.) As a result, the two-point Green function can be written as a sum of two-loop effective diagrams, see Fig. 3.

2. An attempt to present the two-loop effective diagram as an integral of a total derivative encounters considerable problems. The reason can be understood from the results of Ref. [15]. The matter is that the total derivative in this case nontrivially depends on the number of vertexes in a diagram. Therefore, it seems that it is impossible to present the total derivative as an effective diagram. However, the solution can be found. For this purpose we introduce the parameter g according to the following prescription: in the kinetic terms for the matter superfields we make the substitution

$$e^{2V} \to 1 + g(e^{2V}-1); \quad e^{-2V} \to 1 + g(e^{-2V}-1). \tag{13}$$

(It is important that we introduce the parameter g after replacing the argument of the effective action V by θ^4. Therefore, by definition the parameter g will be present only in vertexes containing the internal lines of the gauge superfield.) If $g = 1$, then the action in the generating functional coincides with the $N = 1$ SQED action.

We differentiate the two-point Green function of the gauge superfield with respect to the parameter g. Graphically, the result can be written as a sum of some three-loop effective diagrams (we do not present them here). Our purpose is to present them as an integral of a total derivative. In the coordinate representation an integral of a total derivative can be written as

$$\mathrm{Tr}[x^\mu, \mathrm{Something}], \tag{14}$$

$$\Delta\Gamma^{(2)}_V = \frac{1}{2}\int d^8x\, d^8y\, V_x V_y \frac{\delta^2(\Delta\Gamma)}{\delta V_x \delta V_y} =$$

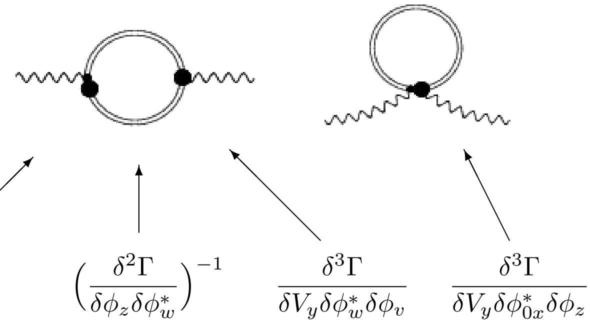

Figure 1: The Schwinger–Dyson equation for the two-point function of the gauge superfield. Below we present Feynman rules (for simplicity, in the massless case). In the massive case the effective diagrams are the same.

where $\mathrm{Tr} \equiv \int d^8x$.

4 Factorization of integrals for β-function into integrals of double total derivatives

The derivative of the two-point Green function of the gauge superfield with respect to $\ln g$ can be written as an integral of a double total derivative. The result can be written in the following form:

$$\frac{d}{d\ln\Lambda}\frac{\partial}{\partial\ln g}\Big(\frac{1}{2}\int d^8x\, d^8y\,(\theta^4)_x(\theta^4)_y\frac{\delta^2\Gamma}{\delta V_x\delta V_y}\Big)$$
$$= \frac{d}{d\ln\Lambda}\Big(\frac{i}{4}\mathrm{Tr}\,(\theta^4)_x\Big[y^*_\mu,\Big[y^*_\mu,\Big(\frac{\delta^2\gamma}{\delta(\phi^{*i})_x\delta(\phi_i)_x}\Big)^{-1}\Big]\Big]$$
$$-\frac{i}{4}\sum_{I=1}^{n} c_I\,\mathrm{Tr}\,(\theta^4)_x\Big[y^*_\mu,\Big[y^*_\mu,\Big(\frac{\delta^2\gamma}{\delta(\Phi^{*i})_x\delta(\Phi_i)_x}\Big)^{-1}$$
$$+M^{ik}\Big(\frac{D^2}{8\partial^2}\Big)_x\Big(\frac{\delta^2\gamma}{\delta(\Phi_i)_x\delta(\Phi_k)_y}\Big)^{-1}_{x=y}$$
$$+M^*_{ik}\Big(\frac{\bar D^2}{8\partial^2}\Big)_x\Big(\frac{\delta^2\gamma}{\delta(\Phi^{*i})_x\delta(\Phi^{*k})_y}\Big)^{-1}_{x=y}\Big]\Big]_I$$
$$-\mathrm{singularities}\Big), \qquad (15)$$

where $\phi_1 \equiv \phi$; $\phi_2 \equiv \tilde\phi$, Φ_i denotes the Pauli–Villars fields, and $(y_\mu)^* \equiv x_\mu - i\bar\theta^a(\gamma_\mu)_a{}^b\theta_b$. In the graphical form the right hand side of identity (15) is presented in Fig. 4. According to Eq. (15) the β-function of the $N=1$ SQED, regularized by higher derivatives, in the momentum representation is given by integrals of double total derivatives.

5 Exact NSVZ β-function

In order to obtain a β-function it is necessary to note that

$$\mathrm{Tr}[y^*_\mu, A] = 0. \qquad (16)$$

In the momentum representation this equation corresponds to vanishing substitution at the upper limit ($q\to\infty$) in the integral of a total derivative. However, the integral does not vanish due to a non-trivial substitution at the lower limit ($q=0$). In the coordinate representation this follows from the existence of singularities [12], which appear in the commutator

$$[x^\mu, \frac{\partial_\mu}{\partial^4}] = [-i\frac{\partial}{\partial p_\mu}, -\frac{ip^\mu}{p^4}] = -2\pi^2\delta^4(p_E)$$
$$= -2\pi^2 i\delta^4(p) = -2\pi^2 i\delta^4(\partial). \qquad (17)$$

Here we take into account that the calculation of loop integrals is made in the Euclidian space after the Wick rotation. A contribution of these singularities with an opposite sign is equal to the sum of diagrams defining the β-function.

Calculating commutators in Eq. (15) it is also possible to find singular contributions. The result can be written as a derivative with respect to $\ln g$. This is a nontrivial test of the calculation. Substituting explicit

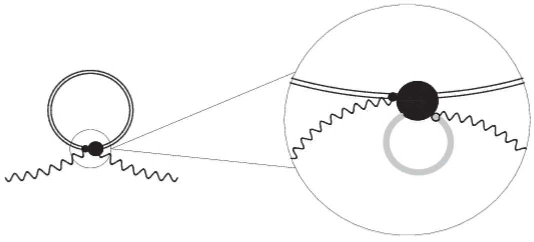

Figure 2: Applying the Schwinger–Dyson equation to the effective vertex it is possible to see "the inner structure" of the effective diagram.

Figure 3: Using the Schwinger–Dyson equations twice we obtain two-loop effective diagrams.

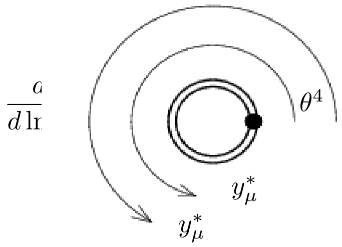

Figure 4: A graphical presentation of double total derivatives given by Eq. (15).

expressions for the Green functions, e.g.

$$\frac{\delta^2\gamma}{\delta(\phi_0)_x\delta(\phi^*)_y} = -\frac{1}{8}GD_x^2\delta_{xy}^8;$$
$$\frac{\delta^2\gamma}{\delta(\phi_0)_x\delta(\phi)_y} = -\frac{1}{32}MJD_x^2\bar{D}_x^2\delta_{xy}^8, \qquad (18)$$

and calculating the integrals over $d^4\theta$, after the Wick rotation in the Euclidean space the final expression can be presented in the following form:

$$\frac{\partial}{\partial\ln g}\frac{d}{d\ln\Lambda}\Big(\frac{1}{2}\int d^8x\,d^8y\,(\theta^4)_x(\theta^4)_y\frac{\delta^2\Gamma}{\delta V_x\delta V_y}\Big)$$
$$= -4\pi^2\delta^4(p=0)\frac{\partial}{\partial\ln g}\frac{d}{d\ln\Lambda}\Big(\ln(q^2G^2) - \sum_{I=1}^{n}c_I$$
$$\times\Big(\ln(q^2G_I^2 + M_I^2J_I^2) + \frac{M_I^2J_I}{(q^2G_I^2+M_I^2J_I^2)}\Big)\Big)_{q=0}. \qquad (19)$$

This result agrees with the calculation made in [17] made by another method. It is easy to see that the contributions of the massive Pauli–Villars fields vanish beyond the one-loop approximation after differentiation with respect to $\ln\Lambda$. Therefore, we obtain

$$\frac{\partial}{\partial\ln g}\frac{d}{d\ln\Lambda}\Big(\frac{1}{2}\int d^8x\,d^8y\,(\theta^4)_x(\theta^4)_y\frac{\delta^2\Gamma}{\delta V_x\delta V_y}\Big)$$
$$= -8\pi^2\delta^4(p=0)\frac{\partial}{\partial\ln g}\frac{d\ln G}{d\ln\Lambda}. \qquad (20)$$

Integrating this equation with respect to $\ln g$ from $g=0$ to $g=1$ and taking into account that $g=0$ corresponds to a theory without quantum gauge superfield, we obtain

$$\frac{\beta(\alpha_0)}{\alpha_0^2} - \frac{\beta(\alpha_0)_{1-loop}}{\alpha_0^2} = -\frac{1}{\pi}\gamma(\alpha_0). \qquad (21)$$

Substituting an explicit expression for the one-loop β-function we obtain NSVZ β-function (1). Note that deriving this result it is not necessary to perform a redefinition of the coupling constant, which is needed if the calculations are made with the dimensional reduction. Therefore the NSVZ-scheme can be naturally constructed using the higher derivative regularization.

6 Conclusion

Using the Schwinger–Dyson equations it is possible to prove that the β-function in the $N=1$ supersymmetric electrodynamics regularized by higher derivatives is given by integrals of double total derivatives in all orders of the perturbation theory. For this purpose we consider the derivative of the two-point Green function of the gauge superfield with respect to the $\ln\Lambda$ in the limit of vanishing external momentum. The Schwinger–Dyson equation for this Green function can be written in terms of two-loop effective diagrams. In order to present the result as a double total derivative it is necessary to differentiate the considered expression with respect to the auxiliary parameter g. The result is given by the effective three-loop diagrams, which can be written as a trace of a double commutator with y_μ^* minus contribution of δ-singularities. In the momentum representation it corresponds to the integrals of double total derivatives. The contribution of the singularities can be related with the anomalous dimension of the matter superfield exactly in all orders of the perturbation theory. This contribution gives the exact NSVZ β-function.

Let us note that the NSVZ β-function with the higher derivative regularization in the considered scheme is obtained without any redefinition of the coupling constant, which is needed if the calculations are made with the dimensional reduction in $\overline{MS}$-scheme. In our case finite counterterms corresponding to the renormalization of α can be arbitrary. Both the β-function and the anomalous dimension depend on them, but the combination $\beta/\alpha^2 + \gamma/\pi$ is invariant. With the higher derivative regularization the finite counterterms corresponding to the renormalization of the matter superfield are fixed by the condition that ratios M_I/Λ are numerical constants and do not depend on the coupling constant. Therefore, using the procedure described in this paper we implicitly fix the subtraction scheme. However, the NSVZ scheme with the higher derivative regularization is obtained in the most natural way.

Acknowledgement

This work was supported by RFBR grant No 11-01-00296a.

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Received 01.10.2012

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ВЫВОД ТОЧНОЙ NSVZ β-ФУНКЦИИ С ПОМОЩЬЮ ЭФФЕКТИВНЫХ ДИАГРАММ

С помощью уравнения Швингера–Дайсона мы доказываем, что β-функция $N = 1$ суперсимметричной электродинамики определяется интегралами от двойныхх полных производных. Это позволяет вычислить один из петелвых интегралов и получить точную NSVZ β-функцию, которая связывает β-функцию и аномальную размерность суперполя материи.

Ключевые слова: *суперсимметрия, суперграфы, NSVZ β-функция.*

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