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# Renormalization of static self-potential

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A method is presented which allows for the renormalization of the self-potential of a scalar point charge at rest in static curved spacetime. The method is based on the local expansion of the self-potential for a scalar point charge at rest in general static spacetimes.

**Keywords:** *self-force, renormalization.*

## 1 Introduction

It is known that a charged particle interacts with the field, the source of which is this particle [1–6]. A discussion of the self-force in detail may be found in reviews [7–9].

Calculating the self-force one must evaluate the field that the point charge induces at the position of the charge. This field diverges and must be renormalized. There are different methods of such type of renormalization. Some of them are reviewed in recent papers [10, 11].

Note also the zeta function method [12] and "the massive field approach" for the calculation of the self-force [13, 14]. In the ultrastatic spacetimes the renormalization of the field of static charge can be realize by the subtraction of the first terms from DeWitt-Schwinger asymptotic expansion of a three dimensional Euclidean Green's function [15–18].

In this paper similar approach expands to the case of static spacetimes. In framework of suggested procedure one subtracts some terms of expansion of the corresponding Green function of a massive scalar field with arbitrary coupling to the scalar curvature from the divergent expression obtained. The quantities of the terms to be subtracted are defined by simple rule – they no longer vanish as the field's mass goes to the infinity.

Such approach is similar to renormalization introduced in the context of the quantum field theory in curved spacetime [19, 20]. The Bunch and Parker method [21] is used for expansion of the corresponding Green's function of a scalar field.

Our conventions are those of Misner, Thorne, and Wheeler [22]. Throughout this paper, we use units $c = G = 1$.

## 2 Renormalization

Let us consider equation for scalar massive field with source

$$\phi_{m;\mu}^{\ ;\mu} - \left(m^2 + \xi R\right)\phi_m = -4\pi q \int \delta^{(4)}(x - x_0(\tau)) \frac{d\tau}{\sqrt{-g^{(4)}}}, \tag{1}$$

where $\xi$ is a coupling of the scalar field with mass $m$ to the scalar curvature $R$, $g^{(4)}$ is the determinant of the metric $g_{\mu\nu}$, $q$ is the scalar charge and $\tau$ is its proper time. The world line of the charge is given by $x_0^\mu(\tau)$. The metric of static spacetime can be presented as follows

$$ds^2 = -g_{tt}(x^i)dt^2 + g_{jk}(x^i)dx^j dx^k, \tag{2}$$

where $i, j, k = 1, 2, 3$. This means that one can write the field equation in the following way

$$\frac{1}{\sqrt{g_{tt}}\sqrt{g^{(3)}}} \frac{\partial}{\partial x^j}\left(\sqrt{g_{tt}}\sqrt{g^{(3)}} g^{jk} \frac{\partial \phi_m(x^i; x_0^i)}{\partial x^k}\right) - \Big(m^2 + \xi R(x)\Big)\phi_m(x^i; x_0^i) = -\frac{4\pi q \delta^{(3)}(x^i, x_0^i)}{\sqrt{g^{(3)}}}, \tag{3}$$

where $m$ is the mass of scalar field, $g^{(3)} = \det g_{ij}$ and we take into account that $d\tau/dt = \sqrt{g_{tt}}$ for the particle at rest. In the case

$$m \gg 1/L, \tag{4}$$

where $L$ is the characteristic curvature scale of the background geometry, it is possible to construct the iterative procedure of the solution of Eq. (3) with small parameter $1/(mL)$ [19–21]. This expansion can be used in the regularization procedure of Rosenthal [13]

$$f_\mu^{self}(x_0) = q \lim_{m\to\infty}\left\{\lim_{\delta\to 0} \frac{\partial\left(\,\phi(x; x_0) - \phi_m(x; x_0)\right)}{\partial x^\mu} + \frac{q m^2 n_\mu(x_0)}{2} + \frac{q m a_\mu(x_0)}{2}\right\}, \tag{5}$$

because this procedure demands the calculation of the expansion of $\phi_m(x;x_0)$ in terms of $x^\mu - x_0^\mu$ and $1/m$ accurate to order $O\left((x-x_0)^2\right)+O\left(1/m\right)$ only. In the expression (5) $\phi(x;x_0)$ is the massless field induced by scalar charge $q$, and $x$ is a point near the charge's word line $x_0(\tau)$, defined as follows. At $x_0$ we construct a unit spatial vector $n^\mu$, which is perpendicular to the object's world line but is otherwise arbitrary (i.e. at $x_0$ we have $n^\mu n_\mu = 1$, $n^\mu u_\mu = 0$). In the direction of this vector we construct a geodesic, which extends out an invariant length $\delta$ to the point $x(x_0, n^\mu, \delta)$; throughout this manuscript $u^\mu$ and $a^\mu$ denote the object's four-velocity and four-acceleration, at $x_0$, respectively.

To construct the expansion of $\phi_m(x;x_0)$, let us consider the equation for the three-dimensional Green's function $G_{\rm E}(x^i, x_0^i)$

$$\frac{1}{\sqrt{g^{(3)}}}\frac{\partial}{\partial x^j}\left(\sqrt{g^{(3)}}g^{jk}\frac{\partial G_{\rm E}(x^i,x_0^i)}{\partial x^k}\right) + \frac{g^{jk}}{2g_{tt}}\frac{\partial g_{tt}}{\partial x^j}\frac{\partial G_{\rm E}(x^i,x_0^i)}{\partial x^k} - \Big(m^2+\xi R(x)\Big)G_{\rm E}(x^i,x_0^i) = -\frac{\delta^{(3)}(x^i,x_0^i)}{\sqrt{g^{(3)}}} \qquad (6)$$

and introduce the Riemann normal coordinates $y^i$ in 3D space with origin at the point $x_0^i$ [23]. In these coordinates one has

$$g_{ij}(y^i) = \delta_{ij} - \frac{1}{3}\widetilde{R}_{ikjl}|_{y=0}\, y^k y^l + O\left(\frac{y^3}{L^3}\right), \qquad (7)$$

$$g^{(3)}(y^i) = 1 - \frac{1}{3}\widetilde{R}_{ij}|_{y=0} y^i y^j + O\left(\frac{y^3}{L^3}\right), \qquad (8)$$

where the coefficients here and below are evaluated at $y^i = 0$ (i.e. at the point $x_0^i$), $\delta_{ij}$ denotes the metric of a flat three-dimensional Euclidean spacetime. $\widetilde{R}_{ikjl}$ and $\widetilde{R}_{ij}$ denote the components of Riemann and Ricci tensors of the three-dimensional spacetime with metric $g_{ij}$

$$R_{ij} = \widetilde{R}_{ij} - \frac{g_{tt,i;j}}{2g_{tt}} + \frac{g_{tt,i}\,g_{tt,j}}{4{g_{tt}}^2},$$
$$R = \widetilde{R} - \frac{{g_{tt,i}}^{;i}}{g_{tt}} + \frac{g_{tt,i}\,{g_{tt}}^{,i}}{2{g_{tt}}^2}, \qquad (9)$$

where $g_{tt,i}$ denotes the covariant derivative of a scalar function $g_{tt}(y^j)$ with respect to $y^i$ in 3D space with metric $g_{ij}(y^k)$ ($g_{tt,i;j}$ is the covariant derivative of a vector $g_{tt,i}$ at point $y^k = 0$ in 3D space, which coincides with partial derivative as $\Gamma^k_{ij} = 0$ at $y^k = 0$ in the Riemann normal coordinates). All indices are raised and lowered with $\delta_{ij}$. Defining $\overline{G}(y^i)$ by

$$\overline{G}(y^i) = \sqrt{g^{(3)}}G_{\rm E}(y^i) \qquad (10)$$

and retaining in (6) only the terms with coefficients involving two derivatives of the metric or fewer one finds that $\overline{G}(y^i)$ satisfies the equation

$$\delta^{ij}\frac{\partial^2\overline{G}}{\partial y^i\partial y^j} - m^2\overline{G} + \delta^{ij}\frac{g_{tt,i}}{2g_{tt}}\frac{\partial\overline{G}}{\partial y^j} + \delta^{ij}\left(\frac{g_{tt,ik}}{2g_{tt}} - \frac{g_{tt,i}\,g_{tt,k}}{2{g_{tt}}^2}\right)y^k\frac{\partial\overline{G}}{\partial y^j} + \widetilde{R}^{\,i\ \ j}_{\ k\ \ l}\frac{y^k y^l}{3}\frac{\partial^2\overline{G}}{\partial y^i\partial y^j} + \left(\frac{\widetilde{R}}{3} - \xi R\right)\overline{G} = -\delta^{(3)}(y). \qquad (11)$$

Let us present

$$\overline{G}(y^i) = \overline{G}_0(y^i) + \overline{G}_1(y^i) + \overline{G}_2(y^i) + \ldots, \qquad (12)$$

where $\overline{G}_a(y^i)$ has a geometrical coefficient involving $a$ derivatives of the metric at point $y^i = 0$. Then these functions satisfy the equations

$$\delta^{ij}\frac{\partial^2\overline{G}_0}{\partial y^i\partial y^j} - m^2\overline{G}_0 = -\delta^{(3)}(y), \qquad (13)$$

$$\delta^{ij}\frac{\partial^2\overline{G}_1}{\partial y^i\partial y^j} - m^2\overline{G}_1 + \delta^{ij}\frac{g_{tt,i}}{2g_{tt}}\frac{\partial\overline{G}_0}{\partial y^j} = 0, \qquad (14)$$

$$\delta^{ij}\frac{\partial^2\overline{G}_2}{\partial y^i\partial y^j} - m^2\overline{G}_2 + \delta^{ij}\frac{g_{tt,i}}{2g_{tt}}\frac{\partial\overline{G}_1}{\partial y^j} + \delta^{ij}\left(\frac{g_{tt,ik}}{2g_{tt}} - \frac{g_{tt,i}\,g_{tt,k}}{2{g_{tt}}^2}\right)y^k\frac{\partial\overline{G}_0}{\partial y^j} + \widetilde{R}^{\,i\ \ j}_{\ k\ \ l}\frac{y^k y^l}{3}\frac{\partial^2\overline{G}_0}{\partial y^i\partial y^j} + \left(\frac{\widetilde{R}}{3} - \xi R\right)\overline{G}_0 = 0. \qquad (15)$$

The function $\overline{G}_0(y^i)$ satisfies the condition

$$\widetilde{R}^{\,i\ \ j}_{\ k\ \ l}\, y^k y^l\frac{\partial^2\overline{G}_0}{\partial y^i\partial y^j} - \widetilde{R}^{\,i}_{\ j} y^j\frac{\partial\overline{G}_0}{\partial y^i} = 0, \qquad (16)$$

since $\overline{G}_0(y^i)$ can be the function only of $\delta_{ij}y^i y^j$. Therefore Eq. (15) may be rewritten

$$\delta^{ij}\frac{\partial^2\overline{G}_2(y^i)}{\partial y^i\partial y^j} - m^2\overline{G}_2(y^i) + \delta^{ij}\frac{g_{tt,i}}{2g_{tt}}\frac{\partial\overline{G}_1}{\partial y^j} + \left[\frac{1}{3}\widetilde{R}^i_k + \delta^{ij}\left(\frac{g_{tt,jk}}{2g_{tt}} - \frac{g_{tt,j}g_{tt,k}}{2{g_{tt}}^2}\right)\right]y^k\frac{\partial\overline{G}_0}{\partial y^i} + \left(\frac{\widetilde{R}}{3} - \xi R\right)\overline{G}_0 = 0. \qquad (17)$$

Let us introduce the local momentum space associated with the point $y^i = 0$ by making the 3-dimensional Fourier transformation

$$\overline{G}_a(y^i) = \iiint\limits_{-\infty}^{+\infty}\frac{dk_1dk_2dk_3}{(2\pi)^3}\exp(ik_iy^i)\overline{G}_a(k^i). \qquad (18)$$

It is not difficult to see that

$$\overline{G}_0(k^i) = \frac{1}{k^2+m^2}, \qquad (19)$$

$$\overline{G}_1(k^i) = i\frac{\delta^{ij} g_{tt,i} k_j}{2g_{tt}(k^2+m^2)^2}, \tag{20}$$

$$\overline{G}_2(k^i) = \frac{-\dfrac{\delta^{ij} g_{tt,ij}}{2g_{tt}} + \dfrac{\delta^{ij} g_{tt,i}\, g_{tt,j}}{2{g_{tt}}^2} - \xi R}{(k^2+m^2)^2}$$
$$+\frac{k_i k_j \delta^{ik}\delta^{jl}\left(\dfrac{2}{3}\widetilde{R}_{j\,l} + \dfrac{g_{tt,j\,l}}{g_{tt}} - \dfrac{5 g_{tt,j}\, g_{tt,l}}{4{g_{tt}}^2}\right)}{(k^2+m^2)^3}, \tag{21}$$

where $k^2 = \delta^{ij} k_i k_j$. Substituting (18), (19), (20), (21) in (12) and integrating leads to

$$\overline{G}_0(y^i) + \overline{G}_1(y^i) + \overline{G}_2(y^i) = \frac{\exp(-my)}{8\pi}\left[\frac{2}{y}\right.$$
$$-\frac{g_{tt,i}}{2g_{tt}}\frac{y^i}{y} + \frac{1}{m}\left(-\frac{\delta^{ij} g_{tt,ij}}{4g_{tt}} + \frac{3\delta^{ij} g_{tt,i}\, g_{tt,j}}{16{g_{tt}}^2} - \xi R\right.$$
$$\left.\left.+\frac{\widetilde{R}}{6}\right) + \left(-\frac{g_{tt,ij}}{4g_{tt}} + \frac{5 g_{tt,i}\, g_{tt,j}}{16{g_{tt}}^2} - \frac{\widetilde{R}_{ij}}{6}\right)\frac{y^i y^j}{y}\right], \tag{22}$$

where

$$y = \sqrt{\delta_{ij} y^i y^j}. \tag{23}$$

Using the definition of $\overline{G}(y^i)$ (10), expansion (8), and expressions (9) one finds

$$G_{\text{E}}(x^i; x_0^i) = \frac{1}{8\pi}\left\{\frac{2}{\sqrt{2\sigma}} + \frac{g_{tt,i}\sigma^i}{2g_{tt}\sqrt{2\sigma}} - 2m\right.$$
$$+\frac{1}{m}\left[-\frac{{g_{tt,i}}^{;i}}{12 g_{tt}} + \frac{5 g_{tt,i}{g_{tt}}^{,i}}{48{g_{tt}}^2} - \left(\xi - \frac{1}{6}\right)R(x_0)\right]$$
$$-m\frac{g_{tt,i}\sigma^i}{2g_{tt}} + m^2\sqrt{2\sigma} + \left[\frac{{g_{tt,i}}^{;i}}{12 g_{tt}} - \frac{5 g_{tt,i}\, {g_{tt}}^{,i}}{48{g_{tt}}^2}\right.$$
$$\left.+\left(\xi - \frac{1}{6}\right)R(x_0)\right]\sqrt{2\sigma} + \left(+\frac{13 g_{tt,i}\, g_{tt,j}}{48{g_{tt}}^2}\right.$$
$$\left.-\frac{g_{tt,i;j}}{6g_{tt}} + \frac{R_{ij}(x_0)}{6}\right)\frac{\sigma^i\sigma^j}{\sqrt{2\sigma}} + O\left(\frac{1}{m^2L^3}\right)$$
$$\left.+O\left(\frac{\sqrt{\sigma}}{mL^3}\right) + O\left(\frac{\sigma}{L^3}\right) + O\left(\frac{m\sigma^{3/2}}{L^3}\right)\right\}, \tag{24}$$

where we take into account that in the arbitrary coordinates of 3D space

$$y^i \to u^i(x_0)\Delta s \equiv -\sigma^i, \tag{25}$$

$u^i(x_0)$ is the unit tangent vector to the shortest geodesic connecting points $x_0$ and $x$ which is calculated at points $x_0$ and directed from $x_0$ to $x$, $\Delta s$ is the distance between these points along the considered geodesic, $g_{tt,i}$ denotes the covariant derivative of a scalar function $g_{tt}(x_0)$ with respect to $x_0^i$ in 3D space with metric $g_{ij}(x_0)$ ($g_{tt,i;j}$ is the covariant derivative of a vector $g_{tt,i}$ at point $x_0$ in 3D space),

$$\sigma = \frac{g_{ij}(x_0)}{2}\sigma^i\sigma^j \tag{26}$$

is one half the square of the distance between the points $x_0^i$ and $x^i$ along the shortest geodesic connecting them and (see, e.g., [24, 25])

$$\sigma^i = -\left(x^i - x_0^i\right) - \frac{1}{2}\Gamma^i_{jk}\left(x^j - x_0^j\right)\left(x^k - x_0^k\right)$$
$$-\frac{1}{6}\left(\Gamma^i_{jm}\Gamma^m_{kl} + \frac{\partial\Gamma^i_{jk}}{\partial x_0^l}\right)\left(x^j - x_0^j\right)\left(x^k - x_0^k\right)$$
$$\left(x^l - x_0^l\right) + O\left((x - x_0)^4\right), \tag{27}$$

where the Christoffel symbols $\Gamma^i_{jk}$ are calculated at the point $x_0$.

Now we can use the expansion of

$$\phi_m(x^i; x_0^i) = 4\pi q G_{\text{E}}(x^i; x_0^i) \tag{28}$$

in the regularization procedure (5). But if we take the limits before the partial differentiation in (5), then the last two terms do not to appear in the expression for $f_\mu^{self}(x_0)$. And in the considered case of a charge at rest in a static spacetime we can renormalize self-potential as

$$\phi_{ren}(x) = \lim_{x_0\to x}\left(\phi(x; x_0) - \phi_{\text{DS}}(x; x_0)\right), \tag{29}$$

where

$$\phi_{\text{DS}} = q\left(\frac{1}{\sqrt{2\sigma}} + \frac{\partial g_{tt}(x_0)}{\partial x_0^i}\frac{\sigma^i}{4g_{tt}(x_0)\sqrt{2\sigma}} - m\right), \tag{30}$$

and $\phi(x; x_0)$ is the solution of (1) in the case of arbitrary mass $m$ (even $m = 0$). Finally the self-force acting on a static scalar charge is

$$f_\mu^{self}(x) = -\frac{q}{2}\frac{\partial\phi_{ren}(x)}{\partial x^\mu}. \tag{31}$$

## 3 Conclusion

The considered approach gives the possibility to renormalize (29) the self-potential of scalar point charge $q$ at rest in static spacetime (2) and to calculate the self-force (31) acting on this charge. Note that in the case the Compton wave length $1/m$ of the massive scalar field is much smaller than the characteristic scale $L$ of curvature of the background gravitational field at the considered point $x$ we can to obtained the approximated expression for the renormalized self-potential

$$\phi_{ren}(x) = \lim_{x_0\to x}\left(\phi_m(x; x_0) - \phi_{\text{DS}}(x; x_0)\right)$$
$$= \frac{q}{2m}\left[-\frac{{g_{tt,i}}^{;i}}{12g_{tt}} + \frac{5g_{tt,i}{g_{tt}}^{,i}}{48{g_{tt}}^2} - \left(\xi - \frac{1}{6}\right)R\right]$$
$$+O\left(\frac{q}{m^2L^3}\right). \tag{32}$$

Of course the order of this expression in $1/(mL)$ is less than the correspondent order of $\phi_{ren}$ for the massless field (or field with mass $m \lesssim 1/L$). However the expression (32) can be used for the verification of asymptotic behavior of $\phi_{ren}$ in the limit $m \to \infty$.

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## ПЕРЕНОРМИРОВКА СТАТИЧЕСКОГО СОБСТВЕННОГО ПОТЕНЦИАЛА

В работе представлен метод перенормировки собственного потенциала скалярного поля, создаваемого покоящимся точечным зарядом, во внешнем статическом гравитационном поле. Метод основан на локальном разложении поля такого заряда в его окрестности.

**Ключевые слова:** *собственный потенциал, перенормировка.*

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