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ENTROPY RELATION OF MULTIPLE HORIZON BLACK HOLES

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New relations for multi-horizon black hole entropy are presented which include entropy product, "part" entropy product and entropy sum. We also discuss their differences and similarities, in order to make a further study on understanding the origin of black hole entropy at the microscopic level with hope.

Keywords: *black hole entropy, entropy sum, partial entropy product.*

1 Introduction

One of the major challenges in quantum theories of gravity in the past years is understanding the origin of black hole entropy at the microscopic level, which is also a clue for probing the microscopics of black holes. For this aim, much attention had been paid to the additional entropy relations of black holes in thermodynamics. These entropy relations include entropy product [1–17], "part" entropy product [11,18] (i. e. $\sum_{1\leq i<j\leq D}(S_iS_j)^{\frac{1}{d-2}}$, where D and d are the number of horizons and the dimensions respectively) and entropy sum [19, 20], which are expected to not only be expressed solely in terms of the quantized charges including the electric charge Q, the angular momentum J, and the cosmological constant Λ (which can be treated as pressure after explaining the mass of the black hole as enthalpy rather than internal energy of the system), but also have the mass independence. These entropy relations are always introduced in black hole spacetime with multi-horizons including the physical horizons: the event horizon, inner (Cauchy) horizon and cosmological horizon; and the un-physical "virtual" horizons.

Consider the thermodynamics of other horizons and the physics inside the black hole is not just an artificial game to play with. There exist several reasons why people study the thermodynamics of all horizons. Firstly, it is found that the Green functions are sensitive to the geometry near all the black hole horizons, and not just the outermost one [21–23]. The thermodynamic properties, especially for the entropy at each horizon, can therefore be expected to play a role in governing the properties of the black hole at the microscopic level. Besides, the parallel study, the entropy inequalities of multi-horizons of four dimensional general axisymmetric stationary solutions in Einstein-Maxwell theory [24–30] are interpreted as a general criterion for extremality [31]. It also results in a No-Go theorem for the possibility of force balance between two rotating black holes [32]. This makes the physics at each horizon more intriguing. On the other hand, in the study on the entropy relations, one need include the necessary effect of the un-physical "virtual" horizons, in order to preserve the mass independence [11, 19, 20]. Only in this way, these additional equalities of multi-horizons of black holes are "universal". Furthermore, by using of these thermodynamical relations, the construction of thermodynamics for inner horizon of black hole catches more attentions [6,7,9,13,14,24,26,28,33], which makes the properties of inner of black hole more interesting.

The study on the entropy product of multi-horizons black hole is generalize to many theories, including the super-gravity model [1–5], Einstein gravity [6–13] and other modified gravity models [3, 14–17] in both four and high dimensions. It is always independent of the mass of the black hole [1–5, 7–10, 12, 16, 17, 19, 20]. However, the mass independence of entropy product fails in some asymptotical non-flat spacetime [6,11,14, 15]. Hence, the "part" entropy produce and entropy sum are introduced, which always are independent of the mass of the black hole in (A)dS spacetime [11, 18–20] (and seem to be more "universal"). To say more accurately, the latter two entropy relations never depend on the electric charge Q and angular momentum J, but the cosmological constant and the constants characterizing the strength of these extra matter field.

In this paper, we firstly present all entropy relations including entropy product, "part" entropy product and entropy sum in three dimensions, which is never studied in previous literature, in order to improve the study on entropy relations. Then we revisit some known entropy relations and give some new unknown ones in four and high dimensions. We discuss their differences and similarities in general dimensions, in order to make a further study on understanding the origin of black hole entropy at the microscopic level. After having a whole look at the entropy relations, we conclude that entropy product and "part" entropy product belong to the same kind of entropy relation, because the mass independence of entropy product and "part" entropy product hold complementary. In rotating (A)dS spacetime, entropy product is independent of mass, while reducing to static (A)dS spacetime "part" entropy product takes the mass independence. When the case in (A)dS spacetime reduces to flat spacetime, it is back to entropy product which is independent of mass, while "part" entropy product turns to depend on mass. Besides, they never holds in the same case. However, there are two kind of failed examples which the mass independence of "part" entropy product or entropy product disappears: the degenerated cases with only two horizons, i.e. the cases that entropy product and "part" entropy product merge into the same entropy product and have some vanishing charges (electric charge and angular momentum, and cosmological constant) meanwhile, such as the static uncharged BTZ black hole and Kerr-Newman black hole [14]; the cases do not admit the entropy area law, such as the three dimensional static uncharged hairy black hole and Gauss-Bonnet-AdS black holes [14]. For the entropy sum, it only depends on the constant characterizing the strength of the background spacetime (the cosmological constant etc.) and never depends on electric charge, angular momentum and mass. Consider the "part" entropy product and entropy sum together, we find that they both have solely the constant (characterizing the strength of the background spacetime, such as the cosmological constant etc.) dependence, other than the electric charge and angular momentum dependence, which is different from the entropy product. It is found that "part" entropy product and entropy sum of Schwarzschild-de-Sitter black hole are actually equal, when only the effect of the physical horizons are considered, as they both can be simplified into a mass independent entropy relations of physical horizon. This also reveal that one can explaining the origin of black hole entropy at the microscopic level without considering the effect of the un-physical virtual horizons. For the sake of brevity, this idea is only checked in Schwarzschild-de-Sitter black hole, while the calculations for the other cases (the Kerr-Newman-(Anti-)de-Sitter black holes etc.) can be performed in a similar manner.

This paper is organized as follows. In the next Section, we will investigate the entropy relations of $2+1$ dimensional black holes. In Section 3, we take as whole look at the entropy relations in four and high dimensions and discuss their differences and similarity in general dimensions. Section 4 is devoted to the conclusions.

2 Entropy relations of black holes with multi-horizons in $(2+1)$ dimensions

Entropy relations including entropy product, "part" entropy product and entropy sum in four and high dimensions are studied widely. In this section, we present all entropy relations of $(2+1)$ dimensional BTZ black hole [34] .

Consider the BTZ black hole [34], which is the solution of Einstein equations in the theory with the lagrangian

$$\mathcal{L} = \frac{1}{2\pi}\int d^3x\sqrt{-g}\,(R-2\Lambda)\,, \tag{1}$$

where the cosmological constant $\Lambda = -\frac{1}{l^2}$. The metric takes the form as

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(N^\phi(r)dt + d\phi)^2,$$

with the horizon function $f(r)$ and the angular velocity $N^\phi(r)$

$$f(r) = -M + \frac{r^2}{\ell^2} + \frac{J^2}{4r^2},$$
$$N^\phi(r) = -\frac{J}{2r^2},$$

where M and J are the mass and angular momentum of the black hole respectively. We are interested in the entropy relations of the multi-horizons, which are the roots of $f(r)$ and read as

$$r_1 = \sqrt{\left(1+\sqrt{1-\left(\frac{J}{M\ell}\right)^2}\right)M\ell},$$
$$r_2 = -\sqrt{\left(1+\sqrt{1-\left(\frac{J}{M\ell}\right)^2}\right)M\ell},$$
$$r_3 = \sqrt{\left(1-\sqrt{1-\left(\frac{J}{M\ell}\right)^2}\right)M\ell},$$
$$r_4 = -\sqrt{\left(1-\sqrt{1-\left(\frac{J}{M\ell}\right)^2}\right)M\ell},$$

where r_1 and r_3 correspond to event horizon and Chauchy horizon, i.e. physical horizons, while r_2 and r_4 represent the negative and un-physical "virtual" horizons which often is discarded in literature.

The entropy at each horizon is equal to twice the perimeter length of the horizon [34], i. e.

$$S_i = 4\pi r_i. \tag{2}$$

A straightforward calculation then gives the entropy relations. The entropy product

$$\prod_{i=1}^{4} S_i = 64\pi^4 J^2 \ell^2, \tag{3}$$

is strictly independent of mass M. It only depends on angular momentum J, which is consistent with the results in general rotating spacetime [1, 3]. We also present the "part" entropy product of BTZ black hole here

$$\sum_{1\le i<j\le 4} S_i S_j = -16\pi^2 M \ell^2. \tag{4}$$

It is mass dependent and not solely cosmological constant dependent, which is different from the case in four and high dimensional static spacetime [18]. The entropy sum

$$\sum_{i=1}^{4} S_i = 0 \tag{5}$$

is vanishing and independent of mass M, and is consistent with the results in general odd dimensions [20].

However, when the J is vanishing, the black hole reduces to static case with only two horizons $r_+ = \sqrt{M}\ell, \quad r_- = -\sqrt{M}\ell$. In this case, the entropy product and "part" entropy product merge into the same entropy product and behavior as a simple relation with two parameters $S_+S_- = -16\pi^2 M\ell^2$, which is mass dependent.

3 Entropy relations of black holes with multi-horizons

In this section, we present all entropy relations by revisiting some known entropy relations and give some new unknown ones in four and high dimensions. We show the entropy product, "part" entropy product and entropy sum in four and high dimensions separately. Then we discuss their differences and similarity in general dimensions.

3.1 Entropy product

The entropy product of multi-horizons black hole is studied widely in many theories, including the supergravity model [1–5], Einstein gravity [6–13] and other modified gravity models [3,14–17] in both four and high dimensions. Hence, we will only summarize its features here.

1. It is always independent of the mass of the black hole, and can be expressed solely in terms of the quantized charges including the electric charge Q, the angular momentum J, and the cosmological constant Λ (which can be treated as pressure after explaining the mass of the black hole as enthalpy rather than internal energy of the system) [1–5, 7–10, 12, 16, 17, 19, 20].
2. It holds for the black holes in four and higher dimensions asymptotically flat and asymptotically (anti-)de Sitter spacetimes, including the ordinary area entropy [6–13] and non-area entropy for which with higher derivative terms in the lagrangian [3, 14–17].
3. One need include the necessary effect of the unphysical "virtual" horizons, in order to preserve its mass independence [11, 19, 20].
4. It is shown that the charge Q, J and Λ plays an important role in this entropy product. When the rotating black holes reduce to static case, the mass independence of entropy product always fails [6, 11, 14]; electric charge Q plays the same role with J in$f(R)$-Maxwell theory, as entropy product of uncharged $f(R)$ black holes depends on mass; in asymptotically flat spacetime, the mass independence of entropy product is destroyed only in the case of two-horizons black holes, even the charge Q and J are not vanishing, such as Kerr-Newman black hole [14].

3.2 "Part" entropy product

In order to preserve the mass independence of entropy relations, "part" entropy product is introduced [11, 18]. The "part" entropy product we have constructed is $\sum_{1\le i<j\le D}(S_iS_j)^{\frac{1}{d-2}}$, where D is the number of horizons, including physical and "virtual" ones; while d is the number of the dimensions. This type of entropy relation was firstly introduced in [11] in four dimensions and then generalized to general dimensions [18]. However, the "part" entropy product could be calculated smoothly only in the Benkenstain-Hawking entropy case. One should note that, for charged black hole in $f(R)$ gravity, we only demonstrate the $d = 4$ case for simplification, since the standard Maxwell energy-momentum tensor is not traceless, which makes people failed to derive higher dimensional black hole/string solutions from $f(R)$ gravity coupled to standard Maxwell field [35–40].

3.3 Entropy sum

Entropy sum [19, 20] is another entropy relation, which is introduced in order to preserve the mass independence of entropy relations. It is shown firstly in four dimensions [19] and is generalized to higher dimensions [20] soon. One should note that, since in

$d = 4$ dimensions, the integration of the GB density $\mathcal{L}_{\text{GB}} = R_{\mu\nu\gamma\delta}R^{\mu\nu\gamma\delta} - 4R_{\mu\nu}R^{\mu\nu} + R^2$ is a topological number and has no dynamics, which makes it out of our discussion. In the high dimensional Einstein-Scalar theory and Einstein-Weyl theory, entropy sum for multi-horizons black holes are still difficult to obtain.

3.4 Differences and similarities between three entropy relations

In this section, we discuss the differences and similarities between entropy product, "part" entropy product and entropy sum in three, four and high dimensions, in order to make a further study on explaining the origin of black hole entropy at the microscopic level.

After having a whole look at the entropy relations, one can conclude that they can be divided into two kinds barely: entropy product and "part" entropy product belong to product relations for entropy, while entropy sum belong to sum relation. For the former relation, one can find that the mass independence of entropy product and "part" entropy product hold complementary. In rotating (A)dS spacetime, entropy product is independent of mass (and depends on electric charge, angular momentum and cosmological constant), while reducing to static (A)dS spacetime "part" entropy product takes the mass independence (and only depends on cosmological constant). When the case in (A)dS spacetime reduces to flat spacetime, it is back to entropy product which is independent of mass, while "part" entropy product turns to depend on mass. Besides, they never holds in the same case. Hence they can be viewed as the same kind of universal entropy relations. Consider the mass independence, it seems that there is always one of the product relations for entropy holding. However, two kind of failed examples are known: the degenerated cases with only two horizons, whose entropy product and "part" entropy product merge into the same entropy product and have some vanishing charges (electric charge and angular momentum, and cosmological constant) meanwhile, such as the static uncharged BTZ black hole and Kerr-Newman black hole [14]; the cases do not admit the entropy area law, such as the three dimensional static uncharged hairy black hole and Gauss-Bonnet-AdS black holes [14]. For the entropy sum, it only depends on the constant characterizing the strength of the background spacetime (the cosmological constant etc.) and never depends on electric charge, angular momentum and mass. Obviously, one can immediately point out the black holes case whose entropy relations having no mass independence, namely the black hole with vanishing electric charge and angular momentum, and cosmological constant. However, for this special case, people will always find only one horizon which is not interesting, such as the Schwarzschild black holes.

4 Conclusion

In this paper, we firstly present all entropy relations include entropy product, "part" entropy product and entropy sum in three dimensions, which is never studied in literature, in order to improve the study on entropy relations. Then we revisit some known entropy relations and give some new unknown ones in four and high dimensions. We discuss their differences and similarity in general dimensions, in order to make a further study on understanding the origin of black hole entropy at the microscopic level. After having a whole look at the entropy relations, we conclude

1. Entropy product and "part" entropy product belong to the same kind of entropy relation, because the mass independence of entropy product and "part" entropy product hold complementary. In rotating (A)dS spacetime, entropy product is independent of mass, while reducing to static (A)dS spacetime "part" entropy product takes the mass independence. When the case in (A)dS spacetime reduces to flat spacetime, it is back to entropy product which is independent of mass, while "part" entropy product turns to depend on mass. Besides, they never holds in the same case.

2. There are two kind of failed examples which the mass independence of "part" entropy product or entropy product disappears: the degenerated cases with only two horizons, i.e. the cases that entropy product and "part" entropy product merge into the same entropy product and have some vanishing charges (electric charge and angular momentum, and cosmological constant) meanwhile, such as the static uncharged BTZ black hole and Kerr-Newman black hole [14]; the cases do not admit the entropy area law, such as the three dimensional static uncharged hairy black hole and Gauss-Bonnet-AdS black holes [14].

3. "Part" entropy product and entropy sum both have solely the constant (characterizing the strength of the background spacetime, such as the cosmological constant etc.) dependence, other than the electric charge and angular momentum dependence, which is different from the entropy product.

4. "Part" entropy product and entropy sum of Schwarzschild-de-Sitter black hole are actually equal, when only the effect of the physical

horizons are considered, as they both can be simplified into a mass independent entropy relations of physical horizon. This also reveal that one can explaining the origin of black hole entropy at the microscopic level without considering the effect of the un-physical virtual horizons.

However, for the sake of brevity, the idea of this paper is only checked in some simplest examples, while the calculations for the other cases can be performed in a similar manner in the future.

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ЭНТРОПИЯ ЧЕРНЫХ ДЫР СО МНОЖЕСТВЕННЫМИ ГОРИЗОНТАМИ

Получены новые соотношения для энтропии черной дыры со множественными горизонтами, включая произведение энтропий, частичное произведение и их сумму. Мы также обсуждаем их различные и общие свойства с целью дальнейших исследований и с надеждой на понимание происхождения энтропии черной дыры на микроскопическом уровне.

Ключевые слова: *черные дыры, энтропия, суммарная энтропия, частичное произведение энтропий.*

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