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Wormholes in 5D models

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We consider our world as a brane embedded in the $5D$ space-time which is a solution of the $5D$ Einstein equations with Λ-term. We do not solve the modified Einstein equations on the brane, instead of this we use exact solution of $5D$ Einstein equations. The energy-momentum tensor appears as the Israel jump condition on the brane. This tensor at the infinity gives positive tension in one side of the wormhole and negative tension in the second part of the wormhole space-time. In the case of the AdS_5 model without Randall-Sundrum asymptotic we found the wormhole metric which satisfies the energy conditions.

Keywords: *wormhole, Randall-Sundrum model, multidimensional theory, energy conditions.*

1 Introduction

The central problem of wormhole physics is the fact that the wormhole's source violates the energy conditions. The exotic matter is a necessary condition for the existence of wormholes [1]. In multidimensional models the $4D$ space-time is understood as a surface embedded in space-time of large dimensions. The Randall-Sundrum showed [2,3] that the Minkowski $4D$ spacetime maybe embedded in the $5D$ anti de Sitter space-time (AdS_5). As was shown in Ref. [4] the gravitational field equations on the $4D$ hypersurface are different from Einstein's equations. Additional terms contain terms quadratic in the stress-energy tensor of matter, as well as a specific term that depends on the bulk. This term is the projection on the brane the $5D$ Weyl tensor and the sign of this term maybe arbitrary. From this point of view, we can expect that wormholes can be the solution of the modified Einstein equations and the exotic matter appears due to the $5D$ Weyl tensor. In this paper we use the exact solutions of 5-dimensional Einstein equations and consider different hypersurfaces with wormhole's metric. In framework of this approach the modified Einstein equations are satisfied automatically and the main problem is the choice of the hypersurface. We consider the simplest embedding in the $5D$ Randall-Sundrum metric [2,3] and black string model metric [5]. We also consider the general case of a static spherically symmetric metric. In all these cases the condition, that the wormhole's hypersurface goes into Randall-Sundrum model's hypersurface far from the wormhole's throat, leads to a violation of the energy conditions. However, if we assume that this condition is violated, in the case of five-dimensional space-time of the Randall-Sundrum model the wormhole's metric can be presented without violation the energy conditions.

2 Randall-Sundrum model

In framework of the Randall-Sundrum model [2,3] the matter is localized in $4D$ hypersurface in AdS_5. The metric,

$$ds^2_{RS} = e^{-2|y|/l}(d\rho^2 + \rho^2 d\Omega^2_{(2)} - dt^2) + dy^2, \qquad (1)$$

is a solution of a $5D$ vacuum Einstein equations

$$G_{AB} = \kappa_5 T_{AB} = \kappa_5(-\Lambda_5 g_{AB} + S_{AB}\delta(y)) \qquad (2)$$

($\Lambda_5 = -\frac{6}{l^2}$) with stress-energy tensor of brane

$$S_{\mu\nu} = -\lambda g_{\mu\nu} \qquad (3)$$

and positive tension λ: $\kappa_5\lambda = +6/l > 0$.

3 Effective 4D Einstein equations

Following the approach suggested in the work [4] the effective $4D$ gravitational field equations in the vacuum Randall-Sundrum brane read

$$G_{\mu\nu} = -\Lambda_4 q_{\mu\nu} + \kappa_4^2\tau_{\mu\nu} + \kappa_5^2\pi_{\mu\nu} - E_{\mu\nu}, \qquad (4)$$

where

$$\pi_{\mu\nu} = -\frac{1}{4}\tau_{\mu\alpha}\tau^{\alpha}_{\nu} + \frac{1}{12}\tau\tau_{\mu\nu} + \frac{1}{8}q_{\mu\nu}\tau_{\alpha\beta}\tau^{\alpha\beta} - \frac{1}{24}q_{\mu\nu}\tau^2, \qquad (5)$$

$\tau_{\mu\nu}$ is the stress-energy tensor of brane, $E_{\mu\nu}$ is the projection of the 5D Weyl tensor on the brane and $\Lambda_4 = \frac{1}{2}\kappa_5\left(\Lambda_5 + \frac{1}{6}\kappa_5\lambda^2\right)$, $\kappa_4^2 = \frac{1}{6}\kappa_5^2\lambda$ are constants.

The sign of projection of the Weyl tensor projection maybe arbitrary and we can assume that wormhole is the solution of modified Einstein equations without violation the energy conditions.

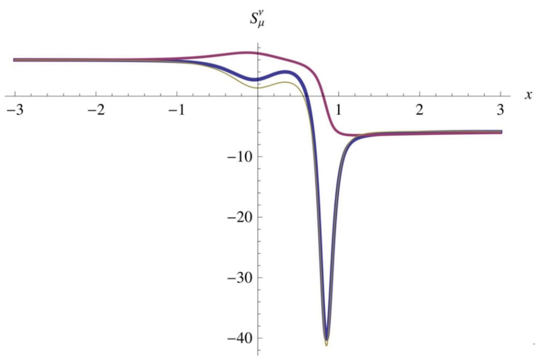

Figure 1: The stress-energy tensor components $S^x_x, S^\theta_\theta, S^t_t$ (thick, medium, thin) for embedding functions of the form $u = \sqrt{x^2+a^2}, v = \frac{l}{2}(1+\tanh^2\frac{x-b}{c})$ $(l = 4, a = 0.5, b = 0.6, c = 0.8)$ as function of the radial coordinate x.

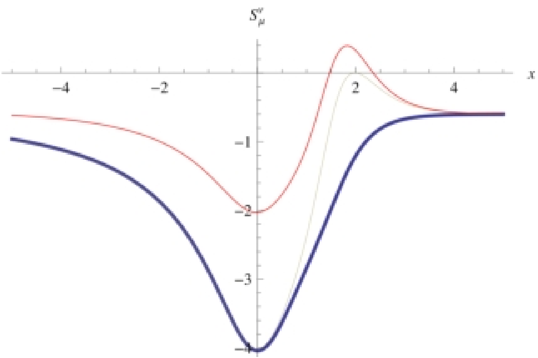

Figure 2: The stress-energy tensor components $S^x_x, S^\theta_\theta, S^t_t$ (thick, medium, thin) for embedding functions of the form $u = \frac{\sqrt{x^2+a^2}}{1+e^{-x/b}}, v = \frac{l}{1+e^{-x/b}}$ $(l = 10, a = 1, b = 0.51)$ as function of the radial coordinate x.

4 Stress-energy tensor of the brane

The brane is delta-like distribution of matter and tension in the bulk. The Israel matching conditions

$$[K^{\mu\nu} - g^{\mu\nu}K] = 8\pi G_5 S^{\mu\nu}, \tag{6}$$

connect the jump of the extrinsic curvature of the hypersurface with stress-energy tensor of the brane matter: $S^{\mu\nu} = -\lambda g^{\mu\nu} + \tau^{\mu\nu}$, where $\tau^{\mu\nu}$ is the stress-energy tensor. Thus the geometry of the brane produces the stress-energy tensor of the matter.

We consider a metric in the form of Randall-Sundrum solution:

$$ds_5^2 = \frac{l^2}{z^2}(d\rho^2 + \rho^2 d\Omega^2_{(2)} - dt^2 + dz^2), \tag{7}$$

where $z > 0, \rho > 0$, and choose the simplest section $z = v(x), \rho = u(x)$ with $x \lessgtr 0$. In this case the metric of the section is

$$ds_4^2 = \frac{l^2}{v^2}\left\{(u'^2+v'^2)dx^2 + u^2 d\Omega^2_{(2)} - dt^2\right\}. \tag{8}$$

Stress-energy tensor of the brane reads

$$S^\mu_\nu = -2(K^\mu_\nu - \delta^\mu_\nu K), \tag{9}$$

or in manifest form

$$\begin{aligned} S^x_x &= -2\frac{3uu'+2vv'}{lu\sqrt{u'^2+v'^2}}, \\ S^\theta_\theta &= -2\frac{vv'(u'^2+v'^2)+u(3u'(u'^2+v'^2)+v(u'v''-v'u''))}{lu(u'^2+v'^2)^{3/2}}, \\ S^t_t &= -2\frac{2vv'(u'^2+v'^2)+u(3u'(u'^2+v'^2)+v(u'v''-v'u''))}{lu(u'^2+v'^2)^{3/2}}. \end{aligned} \tag{10}$$

We impose the asymptotic conditions:

1) RS brane at infinity: $\lim_{x\to\pm\infty} v(x) = c_\pm \neq 0$,

2) Flatness at infinity: $\lim_{x\to\pm\infty} \mathrm{sgn}(u') = \pm 1$.

Then we have the asymptotic value of the stress-energy tensor

$$S^\mu_\nu = -\frac{6}{l}\delta^\mu_\nu\ \mathrm{sgn}(u'). \tag{11}$$

The signs of components of the stress-energy tensor are different on each side of the wormhole's throat in the asymptotic region (Fig. 1). Therefore, if on one side of the wormhole space-time the energy conditions are satisfied then on other side they are violated.

Similarly, we consider section $r = u(x), z = v(x)$ of the Black string space-time [5]

$$ds^2 = \frac{l^2}{z^2}(-U(r)dt^2 + U(r)^{-1}dr^2 + r^2 d\Omega^2_{(2)} + dz^2), \tag{12}$$

where $U(r) = 1 - \frac{2M}{r}$, and spherically symmetric metric of the general form

$$ds^2 = \frac{l^2}{z^2}(e^f dr^2 + r^2 e^h d\Omega^2_{(2)} - e^p dt^2 + e^q dz^2), \tag{13}$$

where $f = f(r,z), h = h(r,z), p = p(r,z), q = q(r,z)$. In both cases requirements in the asymptotic region $\lim_{x\to\pm\infty} v(x) = c_\pm \neq 0, \lim_{x\to\pm\infty} \mathrm{sgn}(u') = \pm 1$ lead to violation of energy conditions, as in the case of the RS metric, $\lim_{x\to\pm\infty} S^\mu_\nu = \mp\frac{6}{l}\delta^\mu_\nu$.

In the case of the RS model, the components of the stress-energy tensor of the brane can have the same signs when $x \to +\infty$ and $x \to -\infty$, if we allow the violation of condition $\lim_{x\to\pm\infty} v(x) = c_\pm \neq 0$. For example this is the case (see Fig. 2) for

$$u = \frac{\sqrt{x^2+a^2}}{1+e^{-x/b}}, \qquad v = \frac{l}{1+e^{-x/b}}. \tag{14}$$

Simple analysis of the geodesics gives some general conclusion. Without loss of generality we choose the plain of motion $\theta = \pi/2$. When $\phi = const$ the first integral of geodesic equations is $\dot{t} = c_4\frac{v^2}{l^2}$. For a massless particle from geodesic equations we obtain the square

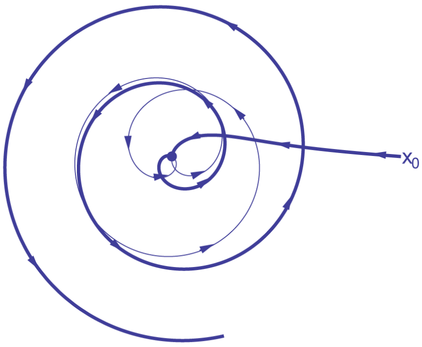

Figure 3: The trajectory of the massive particle in the plain $\theta = \pi/2$. Thick line shows the trajectory in region, where $x(s) > 0$, thin line shows the trajectory in region, where $x(s) < 0$, point indicates the location of the throat, x_0 is the initial location of the particle.

of velocity

$$\dot{x}^2 = \frac{v^4 c_4^2 u^2}{l^4 u^2 (u'^2 + v'^2)}, \tag{15}$$

$$V^2 = \left(\frac{dx}{dt}\right)^2 = \frac{1}{u'^2 + v'^2}. \tag{16}$$

For a massive particle these equations read

$$\dot{x}^2 = \frac{-v^2(-\frac{v^2}{l^2}c_4^2 + 1)}{l^2(u'^2 + v'^2)}, \tag{17}$$

$$V^2 = \left(\frac{dx}{dt}\right)^2 = \frac{-l^2(-\frac{v^2}{l^2}c_4^2 + 1)}{c_4^2 v^2 (u'^2 + v'^2)}. \tag{18}$$

Because the square of the velocity should be positive we obtain the condition

$$x \geq -b\ln(|c_4| - 1). \tag{19}$$

This means, that the particle can not penetrate into specific region of space-time, corresponding to negative values of the x coordinate. When $\phi \neq const$, the trajectory of motion for massive particle is represented on Fig. 3.

5 Conclusion

In this paper we studied the section of the 5D Einstein spacetime with the geometry of a 4D wormhole. It is shown, that the considered sections can not simultaneously satisfy the energy conditions and at the same time coincide with the brane metric in the Randall-Sundrum model in the asymptotic region. We have presented the space-time of the wormhole which corresponds to a brane embedded in the Randall-Sundrum space-time. In this case the matter of the wormhole preserves the energy conditions but the metric asymptotically does not coincide with the brane metric in the Randall-Sundrum model.

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КРОТОВЫЕ НОРЫ В 5D МОДЕЛЯХ

Мы рассматриваем нашу Вселенную как брану, погруженную в пятимерное пространство-время, которое является решением 5D уравнений Эйнштейна с Λ членом. Мы не решаем модифицированных уравнений Эйнштейна на бране, вместо этого мы используем известные решения 5D уравнений Эйнштейна. Тензор энергии-импульса вычисляем с помощью условий сшивки Израэля. Этот тензор на бесконечности дает положительное натяжение с одной стороны от горловины и отрицательное натяжение во второй части пространства-времени кротовой норы. В случае модела AdS_5 (без асимптотики Рандалл-Сундрума) мы нашли метрику кротовой норы, которая удовлетворяет энергетическим условиям.

Ключевые слова: *кротовые норы, многомерные теории, модель Рандалл-Сундрума, браны, энергетические условия.*

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