Instantons and Chern-Simons flows in six and seven dimensions

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1 Introduction

Yang-Mills instantons exist dimensions \(d\) larger than four only when there is additional geometric structure on the manifold \(M^d\) (besides the Riemannian one). In order to formulate generalized first-order anti-self-duality conditions which imply the second-order Yang-Mills equations (possibly with torsion), \(M^d\) must be equipped with a so called \(G\)-structure, which is a globally defined but not necessarily closed \((d-4)\)-form \(\Sigma\), so that the weak holonomy group of \(M^d\) gets reduced.

Instanton solutions in higher dimensions are rare in the literature. In the mid-eighties, Fairlie and Nuyts and also Fubini and Nicolai discovered the \(Sp(7)\)-instanton on \(\mathbb{R}^8\). Eight years later, a similar \(G_2\)-instanton on \(\mathbb{R}^7\) was found by Ivanova and Popov and also by Günaydin and Nicolai. Our recent work shows that these so called octonionic instantons are not isolated but embedded into a whole family living on a class of conical non-compact manifolds [1].

The string vacua in heterotic flux compactifications contain non-abelian gauge fields which in the supergravity limit are subject to Yang-Mills equations with torsion \(\mathcal{H}\) determined by the three-form flux. Prominent cases admitting instantons are \(AdS_{10-d} \times M^d\), where \(M^d\) is equipped with a \(G\)-structure, with \(G\) being \(SU(3)\), \(G_2\) or \(Spin(7)\) for \(d = 6\), 7 or 8, respectively. Homogeneous nearly Kähler 6-manifolds \(\mathbb{R}^7\) and (iterated) cylinders and (sine-)cones over them provide simple examples, for which all \(K\)-equivariant Yang-Mills connections can be constructed [2,3]. Natural choices for the gauge group are \(K\) or \(G\).

Clearly, the Yang-Mills instantons discussed here serve to construct heterotic string solitons, as was first done in 1990 by Strominger for the gauge five-brane. It is therefore of interest to extend our new instantons to solutions of (string-corrected) heterotic supergravity and obtain novel string/brane vacua [4-6].

In this talk, I present the construction for the simplest case of a cylinder over a compact homogeneous nearly Kähler coset \(\frac{\mathrm{U}(1)}{\mathbb{Z}_2}\), which allows for a conformally parallel or a calibrated \(G_2\)-structure. I display a family of non-BPS Yang-Mills connections, which contain two instantons at distinguished parameter values corresponding to those \(G_2\)-structures. In these two cases, anti-self-duality implies a Chern-Simons flow on \(\frac{\mathrm{K}}{\mathbb{Z}_2}\).

Finally, I must apologize for the omission – due to page limitation – of all relevant literature besides my own papers on which this talk is based. The reader can find all references therein.

2 Self-duality in higher dimensions

The familiar four-dimensional anti-self-duality condition for Yang-Mills fields \(F\) may be generalized to suitable \(d\)-dimensional Riemannian manifolds \(M\),

\[
\ast F = - \Sigma \wedge F \quad \text{with} \quad \Sigma \in \Lambda^{d-4}(M)
\]

for \(F = dA + A \wedge A \in \Lambda^2(M)\),

if there exists a geometrically natural \((d-4)\)-form \(\Sigma\) on \(M\). Applying the gauge-covariant derivative \(D = d + [A,\cdot]\) it follows that

\[
D\ast F + d\Sigma \wedge F = 0 \quad \Leftrightarrow \quad \text{Yang-Mills with torsion} \quad \mathcal{H} = \ast d\Sigma \in \Lambda^3(M).
\]

This torsionful Yang-Mills equation extremizes the action

\[
\text{SYM} + \text{CS} = \int_M \text{tr}\left\{ F \wedge \ast F + (-)^{d-3} \Sigma \wedge F \wedge F \right\}
\]

\[
= \int_M \text{tr}\left\{ F \wedge \ast F + \frac{1}{2} d\Sigma \wedge (A dA + \frac{2}{3} A^3) \right\}.
\]
Related to this generalized anti-self-duality is the gradient Chern-Simons flow on $M$,
\[
\frac{d \xi}{dt} = \frac{\delta}{\delta \xi} \text{CS}_1 = \ast (d \xi \wedge F) \sim \ast d \xi \wedge F. \quad (4)
\]

In fact, this equation follows from generalized anti-self-duality on the cylinder $M = \mathbb{R}_x \times \mathbb{M}$ over $M$ (in the $A_r = 0$ gauge).

The question is therefore: Which manifolds admit a global $(d-4)$-form? And the answer is: $G$-structure manifolds, i.e. manifolds with a weak special holonomy. Some of the key cases we shall encounter are listed in Table 1. For this talk I shall consider (reductive nonsymmetric) coset spaces $M = K/H$ in $d = 6$ as well as cylinders and cones over them. In all these cases, the gauge group is chosen to be $K$.

### 3 Six dimensions: nearly Kähler coset spaces

All known compact nearly Kähler 6-manifolds $M^6$ are nonsymmetric coset spaces $K/H$:
\[
S^6 = \frac{\text{SU}(3)}{\text{Sp}(2) / \text{Sp}(1) \times \text{U}(1)}, \quad \frac{\text{SU}(3)}{\text{SU}(1) \times \text{U}(1)}; \quad \text{SU}(2) \times \text{SU}(2). \quad (5)
\]

The coset structure $H < K$ implies the decomposition
\[
\text{Lie}(K) \equiv \mathfrak{h} \oplus \mathfrak{m}, \quad \mathfrak{h} \equiv \text{Lie}(H), \quad \mathfrak{h} \cap \mathfrak{m} \subset \mathfrak{m}. \quad (6)
\]

Interestingly, the reflection automorphism of symmetric spaces gets generalized to a so called tri-symmetry automorphism $S : K \to K$ with $S^3 = \text{id}$ implying $s : \mathfrak{h} \to \mathfrak{h}$ with
\[
s|_{\mathfrak{h}} = \mathbb{I} \quad \text{and} \quad s|_{\mathfrak{m}} = -\frac{1}{2} + \frac{\sqrt{3}}{2} J = \exp\left(\frac{\sqrt{3}}{2} J\right), \quad (7)
\]
efflicting a $2\frac{\pi}{3}$ rotation on $TM^6$. I pick a Lie-algebra basis
\[
\{I_{a=1,\ldots,6}, \ I_{i=7,\ldots,\dim G}\}, \quad \{I_a, I_b\} = f_{ab} I_a I_b + f_{ab} I_c, \quad (8)
\]
involving the structure constants $f_{ab}$. The Cartan-Killing form then reads
\[
\langle \cdot, \cdot \rangle_k = -\text{tr}_{\mathfrak{g}} (\text{ad} (\cdot) \circ \text{ad} (\cdot)) = 3 \langle \cdot, \cdot \rangle_h = 3 \langle \cdot, \cdot \rangle_m = \mathbb{I}. \quad (9)
\]

Expanding all structures in a basis of canonical one-forms $e^a$ framing $T^*(G/H)$,
\[
g = \delta_{ab} e^a e^b, \quad \omega = \frac{1}{2} J_{ab} e^a \wedge e^b, \quad (10)
\]
\[
\Omega = -\frac{1}{\sqrt{3}} (f + iJ)_{abc} e^a \wedge e^b \wedge e^c,
\]
we see that the almost complex structure $(J_{ab})$ and the structure constants $f_{abc}$ rule everything.

Nearly Kähler 6-manifolds are special in that the torsion term in (2) vanishes by itself! What is more, this property is actually equivalent to the generalized anti-self-duality condition (1):
\[
\ast F = -\omega \wedge F \iff 0 = d\omega \wedge F \sim \text{Im} \Omega \wedge F \iff \text{DUY equations}, \quad (11)
\]
where the Donaldson-Uhlenbeck-Yau (DUY) equations\(^1\) state that
\[
F^{2,0} = F^{0,2} = 0 \quad \text{and} \quad \omega_{,F} = 0. \quad (12)
\]
Another interpretation of this anti-self-duality condition is that is projects $F$ to the 8-dimensional eigenspace of the endomorphism $\ast (\omega \wedge \cdot)$ with eigenvalue $-1$, which contains the part of $F^{1,1}$ orthogonal to $\omega$. The equations (11) imply also $\text{Re} \Omega \wedge F = 0$ and the (torsion-free) Yang-Mills equations $D_* F = 0$.

Clearly, they separately extremize both $S_{\text{YM}}$ and $S_{\text{CS}}$ in (3), but of course yield only BPS-type classical solutions. In components the above relations take the form
\[
\frac{1}{4} f_{abcd} e F_{ef} = -J_{[ab} F_{cd]} \iff 0 = f_{abc} F_{bc}, \quad (13)
\]
\[
\omega_{ab} F_{ab} = 0, \quad (JF)_{abc} F_{bc} = 0, \quad D_{ab} F_{ab} = 0. \quad (14)
\]

I notice that each Chern-Simons flow $\dot{A}_a \sim f_{abc} F_{bc}$ on $M^6$ ends in an instanton.

Let me look for $K$-equivariant connections $A$ on $M^6$. If I restrict their value to $\mathfrak{h}$, the answer is

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\(^1\)also known as 'hermitian Yang-Mills equations'
unique: the only ‘H-instanton’ is the so called canonical connection
\[ A^{\text{can}} = e^i I_i ] \Rightarrow = F^{\text{can}} = -\frac{1}{2} f^i_{ab} e^a \wedge e^b I_i \],
(15)
where \( e^i = e^i I_i \). Generalizing to ‘K-instantons’, I extend to the ansatz
\[ A = e^i I_i + e^a \Phi_{ab} I_b \quad \text{with} \quad (\Phi_{ab}) = :\Phi = \phi I + \phi_2 J ,
(16)
with one complex parameter \( \phi = \phi_1 + i \phi_2 \). This is in fact general for \( G_2 \) invariance on \( S^6 \). Its curvature is readily computed to
\[ F_{ab} = F^{1,1}_{ab} + F^{2,0\rightarrow 0,2}_{ab} = ([\Phi]^2 - f^a_{ab} I_i + [(\Phi^2 - \Phi)]_{abc} I_c \]
(17)
and displays the tri-symmetry under \( \Phi \rightarrow \exp[\frac{2\pi i}{3}] \Phi \).
The solutions to the BPS conditions (11) are
\[ \Phi^2 = \Phi \quad \Rightarrow \quad \Phi = 0 \quad \text{or} \quad \Phi = \exp[\frac{2\pi i}{3} J] \]
(18)
for \( k = 0, 1, 2 \), which yields three flat \( K \)-instanton connections besides the canonical curved one,
\[ A^{(k)} = e^i I_i + e^a (s^k l) a \quad \text{and} \quad A^{\text{can}} = e^i I_i . \]
(19)

4 Seven dimensions: cylinder over nearly Kähler

Let me step up one dimension and consider 7-manifolds \( M^7 \) with weak \( G_2 \) holonomy associated with a \( G_2 \)-structure three-form \( \psi \). Here, the 7-generalized anti-self-duality equations project \( F \) onto the \(-1\) eigenspace of \( \ast (\psi \wedge \cdot ) \), which is 14-dimensional and isomorphic to the Lie algebra of \( G_2 \),
\[ \ast F = - \psi \wedge F \iff \ast \psi \wedge F = 0 \iff \psi \wedge F = 0 , \]
(20)
providing an alternative form of the condition. In components, it reads
\[ \frac{1}{2} \varepsilon_{abcdefg} F_{fg} = - \psi_{[abc} F_{dce]} \iff 0 = \psi_{abc} F_{dce} . \]
(21)
For the parallel and nearly parallel \( G_2 \) cases, the previous accident (11) recurs,
\[ d\psi \sim \ast \psi \quad \Rightarrow \quad d\psi \wedge F = 0 \quad \Rightarrow \quad D^* F = 0 , \]
(22)
and the torsion decouples. Note that on a general weak \( G_2 \)-manifold there are two different flows \( (\sigma \in \mathbb{R}) \),
\[ \frac{dA}{d\sigma} = \ast d\psi \wedge F(\sigma) \quad \text{and} \quad \frac{dA}{d\sigma} = \psi \wedge F(\sigma) , \]
(23)
which coincide in the nearly parallel case. The second flow ends in an instanton on \( M^7 \).

In this talk I focus on cylinders \( M^7 = \mathbb{R} \times K/\tilde{H} \) over nearly Kähler cosets, with a metric \( g = (d\tau)^2 + \delta_{ab} e^a e^b \). Note that these do not carry a parallel or nearly parallel \( G_2 \) structure but admit a conformally parallel one. I will study the Yang-Mills equation with a torsion given by
\[ \ast H = \frac{1}{2} \kappa (d \omega \wedge d\tau \iff T_{abc} = \kappa f_{abc} \]
(24)
with a real parameter \( \kappa \). We shall see that for special values of \( \kappa \) my torsionful Yang-Mills equation
\[ D^* F + \frac{1}{2} \kappa (d \omega \wedge d\tau \wedge F = 0 \]
(25)
descends from an anti-self-duality condition (20).

Taking the \( A_0 = 0 \) gauge and borrowing the ansatz (16) from the nearly Kähler base, I write
\[ A_a = e^i I_i + [\Phi(\tau) I] \quad \Rightarrow \quad F_{ab} = [\Phi I]_{ab} \]
(26)
and
\[ F_{ab} = ([\Phi]^2 - 1) f^a_{ab} I_i + [(\Phi^2 - \Phi)]_{abc} I_c \]
(27)
which depends on a function \( \phi(\tau) = \phi_1(\tau) + i \phi_2(\tau) \in \mathbb{C} \). Sticking this into (25) and computing for a while, one arrives at
\[ \phi = (\kappa - 1) \phi - (\kappa + 3) \phi^2 + 4 \phi \phi^2 =: \frac{1}{3} \frac{\partial V}{\partial \phi} . \]
(28)
Nicely enough, I have obtained a \( \phi^4 \) model with an action
\[ S[\phi] = \int d\tau \{ 3 |\phi|^2 + V(\phi) \} \quad \text{for} \quad V(\phi) = (3 - \kappa) + 3(\kappa - 1)|\phi|^2 - 3(\kappa + 3)|\phi^3 + \phi^3|^2 + 6|\phi|^4 \]
(29)
devoid of rotational symmetry (for \( \kappa \neq -3 \)) but enjoying tri-symmetry in the complex plane. It leads me to a mechanical analog problem of a Newtonian particle on \( S^2 \) in a potential \(-V \). I obtain the same action by plugging (26) directly into (3) with \( d\Sigma = \ast H \) from (24).

For the case of \( K = S^0 = \mathbb{G}_2 / S(\mathbb{C}) \), equation (27) produces in fact all \( G \)-equivariant Yang-Mills connections on \( \mathbb{R} \times K/\tilde{H} \). On \( \mathfrak{sp}(2) \) and \( \mathfrak{su}(3) \), however, the most general \( G \)-equivariant connections involves two respective three complex functions of \( \tau \). The corresponding Newtonian dynamics on \( \mathbb{C}^2 \) respective \( \mathbb{C}^3 \) is of similar type but constrained by the conservation of Noether charges related to relative phase rotations of the complex functions [3].
5 Seven dimensions: solutions

Finite-action solutions require Newtonian trajectories between zero-potential critical points $\phi$. With two exotic exceptions, $dV(\hat{\phi}) = 0 = V(\hat{\phi})$ yields precisely the BPS configurations on $K_H$:

- $\hat{\phi} = e^{\pm 2\pi i k/3}$ with $V(\hat{\phi}) = 0$ for all values of $\kappa$ and $k = 0, 1, 2$
- $\hat{\phi} = 0$ with $V(\hat{\phi}) = 3 - \kappa$ vanishing only at $\kappa = 3$

Kink solutions will interpolate between two different critical points, while bounces will return to the critical starting point. Thus for generic $\kappa$ values one may have kinks of ‘transversal’ type, connecting two third roots of unity, as well as bounces. For $\kappa = 3$ ‘radial’ kinks, reaching such a root from the origin, may occur as well. Numerical analysis reveals the domains of existence in $\kappa$:

<table>
<thead>
<tr>
<th>$\kappa$ interval</th>
<th>types of trajectory</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(-\infty, -3]$</td>
<td>radial</td>
</tr>
<tr>
<td>$(-3, +3)$</td>
<td>transversal</td>
</tr>
<tr>
<td>$+3$</td>
<td>radial</td>
</tr>
<tr>
<td>$+3, +5$</td>
<td>radial</td>
</tr>
</tbody>
</table>

Magic happens at three special values of $\kappa$: At $\kappa = -3$ rotational symmetry emerges; this is a degenerate situation. At $\kappa = -1$ and at $\kappa = +3$, displayed in Figure 1 above, the kink trajectories are straight lines, indicating integrability. Indeed, behind each of these two cases lurks a first-order flow equation, which originates from anti-self-duality and hence a particular $G_2$-structure $\psi$.

Let me first discuss $\kappa = +3$. For this value I find that

$$3\ddot{\phi} + \frac{\partial V}{\partial \phi} = \pm \sqrt{2}\dot{\phi} = \pm \frac{\partial W}{\partial \bar{\phi}}$$

with

$$W = \frac{1}{3}(\phi^3 + \bar{\phi}^3) - |\phi|^2 \quad \text{for} \quad \kappa = +3. \quad (29)$$

It admits the obvious analytic radial kink solution,

$$\phi(\tau) = e^{2\pi i (\frac{1}{2} \pm \frac{1}{2} \tanh \frac{\tau}{2\sqrt{3}})} \quad \text{for} \quad \kappa = 0, 1, 2. \quad (30)$$

What is the interpretation of this gradient flow in terms of the original Yang-Mills theory? Demanding that the torsion in (24) comes from a $G_2$-structure, $\ast H = d\psi$, I am led to

$$\psi = \frac{1}{2}\omega \wedge d\tau + \alpha \text{Im} \Omega \quad \Rightarrow \quad d\psi \sim \kappa \text{Im} \Omega \wedge d\tau \sim \psi \wedge d\tau$$

where $\alpha$ is undetermined. This is a conformally parallel $G_2$-structure, and (20) quantizes the coefficients to $\alpha = 1$ and $\kappa = 3$, fixing (with $e^\tau = r$)

$$\psi = \omega \wedge d\tau + \text{Im} \Omega = \frac{1}{4}(r^2 \omega \wedge d\tau + r^3 \text{Im} \Omega) = \frac{1}{r^2} \psi_{\text{cone}} \quad \text{for} \quad \kappa = +3. \quad (31)$$

where I displayed the conformal relation to the parallel $G_2$-structure on the cone over $K_H$.

Alternatively, with this $G_2$-structure the 7 anti-self-duality equations (20) turn into

$$\omega \llcorner F \sim J_{ab} F_{ab} = 0 \quad \text{and} \quad 3\dot{\omega} \llcorner F \sim \epsilon^a f_{abc} F_{bc} \quad \text{for} \quad \kappa = +3. \quad (32)$$
With the ansatz (26), the first relation is automatic, and the second one indeed reduces to (29). As a consistency check, one may verify that

$$\int_{\phi} \text{tr} \{ \omega \wedge F \wedge F \} \propto W(\phi) + \frac{1}{4} \cdot$$ (35)

I now come to the other instance of straight trajectories, \( \kappa = -1 \). For this value I find that

$$3\phi = \frac{\partial V}{\partial \phi} \quad \Leftrightarrow \quad \sqrt{2}\phi = \pm i \frac{\partial W}{\partial \phi}$$ (36)

with \( H = \frac{1}{4}(\phi^3 + \bar{\phi}^3) - |\phi|^2 \),

which is a Hamiltonian flow (note the imaginary multiplier), running along the level curves of the function \( H \), that is identical to \( W \). It has the obvious analytic transverse kink solution,

$$\phi(\tau) = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i (\tanh \frac{\tau}{2})$$ (37)

and its images under the tri-symmetry action.

Have I discovered another hidden \( G_2 \)-structure here? Let me try the other obvious choice,

$$\tilde{\psi} = \frac{\kappa}{2} \omega \wedge d\tau + \bar{\alpha} \text{Re} \Omega \quad \Rightarrow \quad d\tilde{\psi} \sim \tilde{\kappa} \text{Im} \Omega \wedge d\tau + 2\bar{\alpha} \omega \wedge \omega$$, (38)

with coefficients \( \tilde{\kappa} \) and \( \tilde{\alpha} \) to be determined. It has not appeared in my table in Section 2, but obeys \( d^* \tilde{\psi} = 0 \), which is known as a co-calibrated \( G_2 \)-structure. But can it produce the proper torsion,

$$d\tilde{\psi} \wedge F \sim (\tilde{\kappa} \text{Im} \Omega \wedge d\tau + 2\bar{\alpha} \omega \wedge \omega) \wedge F \quad \Rightarrow \quad -\text{Im} \Omega \wedge d\tau \wedge F \ ?$$ (39)

Employing the anti-self-duality with respect to \( \tilde{\psi} \),

$$\ast \tilde{\psi} \wedge F = 0 \quad \Rightarrow \quad \omega \wedge \omega \wedge F = 2 \text{Im} \Omega \wedge d\tau \wedge F \ , \ (40)$$

it works out, adjusting the coefficients to \( \tilde{\kappa} = 3 \) and \( \bar{\alpha} = -1 \). Hence, the co-calibrated \( G_2 \)-structure

$$\tilde{\psi} = \omega \wedge d\tau - \text{Re} \Omega$$ (41)

is responsible for the Hamiltonian flow.

To see this directly, I import (41) into (20) and get

$$J_{ab}F_{bc} = 0 \quad \text{and} \quad \hat{A}_a \sim [Jf]_{abc}F_{bc} \ . \ (42)$$

Again, the ansatz (26) fulfills the first relation, and the second one nicely turns into (36).

6 Partial summary

Let me schematically sum up the construction.

$$\psi \wedge F = - * F \quad \text{on} \quad \mathbb{R} \times \frac{K}{H} \quad \tilde{\psi} \wedge F = - * F$$

$$\hat{A}_a \sim f_{abc}F_{bc} \quad \hat{A}_a \sim [Jf]_{abc}F_{bc}$$

$$\sqrt{2}\phi = \pm i \frac{\partial W}{\partial \phi} \quad \sqrt{2}\phi = \pm i \frac{\partial H}{\partial \phi}$$

$$W = \frac{1}{3}(\phi^3 + \bar{\phi}^3) - |\phi|^2 = H$$

$$F(\tau) = d\tau \wedge c^a [\Phi I]_a + \frac{1}{3} c^a \wedge c^b \{([\Phi]^2-1) f_{ab} I_i + ([\Phi^2-\Phi] f)_{abc} I_c \}$$

are \( G_2 \)-instantons for Yang-Mills with torsion \( D^*F + (*H) \wedge F = 0 \)

from

$$S[A] = \int_{\mathbb{R} \times \frac{K}{H}} \text{tr} \{ F \wedge * F + \frac{1}{3} \kappa \omega \wedge d\tau \wedge F \wedge F \} \quad \text{with} \quad \kappa = +3 \text{ or } -1$$

and obey gradient or Hamiltonian flow equations for

$$\int_{\frac{K}{H}} \text{tr} \{ \omega \wedge F \wedge F \} \propto W(\phi) + \frac{1}{3} \ .$$

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