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Instantons and Chern-Simons flows in six and seven dimensions

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The existence of K-instantons on a cylinder $M^7=\mathbb{R}_\tau\times\frac{K}{H}$ over a homogeneous nearly Kähler 6-manifold $\frac{K}{H}$ requires a conformally parallel or a cocalibrated G_2-structure on M^7. The generalized anti-self-duality on M^7 implies a Chern-Simons flow on $\frac{K}{H}$ which runs between instantons on the coset. For K-equivariant connections, the torsionful Yang-Mills equation reduces to a particular quartic dynamics for a Newtonian particle on $\mathbb{C}$. We obtain kink- or bounce-type solutions for generic values of the torsion. When the latter corresponds to the conformally parallel or cocalibrated G_2-structure on M^7, the dynamics follows from a gradient or hamiltonian flow, respectively, and we encounter Yang-Mills instantons.

Keywords: *instantons, Chern-Simons flow, special geometry, G-structures, nearly-Kähler manifolds.*

1 Introduction

Yang-Mills instantons exist dimensions d larger than four only when there is additional geometric structure on the manifold M^d (besides the Riemannian one). In order to formulate generalized first-order anti-self-duality conditions which imply the second-order Yang-Mills equations (possibly with torsion), M^d must be equipped with a so called G-structure, which is a globally defined but not necessarily closed $(d{-}4)$-form Σ, so that the weak holonomy group of M^d gets reduced.

Instanton solutions in higher dimensions are rare in the literature. In the mid-eighties, Fairlie and Nuyts and also Fubini and Nicolai discovered the Spin(7)-instanton on $\mathbb{R}^8$. Eight years later, a similar G_2-instanton on $\mathbb{R}^7$ was found by Ivanova and Popov and also by Günaydin and Nicolai. Our recent work shows that these so called octonionic instantons are not isolated but embedded into a whole family living on a class of conical non-compact manifolds [1].

The string vacua in heterotic flux compactifications contain non-abelian gauge fields which in the supergravity limit are subject to Yang-Mills equations with torsion $\mathcal{H}$ determined by the three-form flux. Prominent cases admitting instantons are $\mathrm{AdS}_{10-d}\times M^d$, where M^d is equipped with a G-structure, with G being SU(3), G_2 or Spin(7) for $d=6$, 7 or 8, respectively. Homogeneous nearly Kähler 6-manifolds $\frac{K}{H}$ and (iterated) cylinders and (sine-)cones over them provide simple examples, for which all K-equivariant Yang-Mills connections can be constructed [2, 3]. Natural choices for the gauge group are K or G.

Clearly, the Yang-Mills instantons discussed here serve to construct heterotic string solitons, as was first done in 1990 by Strominger for the gauge five-brane. It is therefore of interest to extend our new instantons to solutions of (string-corrected) heterotic supergravity and obtain novel string/brane vacua [4–6].

In this talk, I present the construction for the simplest case of a cylinder over a compact homogeneous nearly Kähler coset $\frac{K}{H}$, which allows for a conformally parallel or a cocalibrated G_2-structure. I display a family of non-BPS Yang-Mills connections, which contain two instantons at distinguished parameter values corresponding to those G_2-structures. In these two cases, anti-self-duality implies a Chern-Simons flow on $\frac{K}{H}$.

Finally, I must apologize for the omission – due to page limitation – of all relevant literature besides my own papers on which this talk is based. The reader can find all references therein.

2 Self-duality in higher dimensions

The familiar four-dimensional anti-self-duality condition for Yang-Mills fields F may be generalized to suitable d-dimensional Riemannian manifolds M,

$$
\begin{aligned}
*F \;=\; -\Sigma\wedge F \qquad &\text{with} \qquad \Sigma\in\Lambda^{d-4}(M)\\
\text{for} \qquad & F=\mathrm{d}A+A\wedge A\in\Lambda^2(M)\ ,
\end{aligned}
\tag{1}
$$

if there exists a geometrically natural $(d{-}4)$-form Σ on M. Applying the gauge-covariant derivative $D=\mathrm{d}+[A,\cdot]$ it follows that

$$
\begin{aligned}
D{*}F \;+\; \mathrm{d}\Sigma\wedge F \;=\; 0 \qquad &\Leftrightarrow\\
\text{Yang-Mills with torsion} \quad \mathcal{H}={*}\mathrm{d}\Sigma\in\Lambda^3(M)\ .&
\end{aligned}
\tag{2}
$$

This torsionful Yang-Mills equation extremizes the action

$$
\begin{aligned}
S_{\mathrm{YM}}+S_{\mathrm{CS}} \;&=\; \int_M \mathrm{tr}\big\{F\wedge{*}F \;+\; (-)^{d-3}\Sigma\wedge F\wedge F\big\}\\
&=\; \int_M \mathrm{tr}\big\{F\wedge{*}F \;+\; \tfrac12\mathrm{d}\Sigma\wedge\big(A\,\mathrm{d}A+\tfrac23A^3\big)\big\}\ .
\end{aligned}
\tag{3}
$$

d	G	Σ	cases	example	structure
6	SU(3)	ω	Kähler	$\mathbb{C}P^3$	$\mathrm{d}\omega=0$
6	SU(3)	ω	nearly Kähler	$S^6=\frac{G_2}{\mathrm{SU}(3)}$	$\mathrm{d}\omega\sim\mathrm{Im}\Omega\,,\ \mathrm{dRe}\Omega\sim\omega^2$
7	G_2	ψ	conf. parallel G_2	$\mathbb{R}_\tau\times$ nearly Kähler	$\mathrm{d}\psi\sim\psi\wedge\mathrm{d}\tau\,,\ \mathrm{d}{*}\psi\sim-{*}\psi\wedge\mathrm{d}\tau$
7	G_2	ψ	nearly parallel G_2	$X_{k,\ell}=\frac{\mathrm{SU}(3)}{\mathrm{U}(1)_{k,\ell}}$	$\mathrm{d}\psi\sim{*}\psi\ \Rightarrow\ \mathrm{d}{*}\psi=0$
7	G_2	ψ	parallel G_2	cone(nearly Kähler)	$\mathrm{d}\psi=0=\mathrm{d}{*}\psi$
8	Spin(7)	Σ	parallel Spin(7)	$\mathbb{R}_\tau\times$ parallel G_2	$\mathrm{d}\Sigma=0\,,\ {*}\Sigma=\Sigma$

Table 1: Examples of G-structure manifolds at $d=6,7,8$.

Related to this generalized anti-self-duality is the gradient Chern-Simons flow on M,

$$\frac{\mathrm{d}A}{\mathrm{d}\tau}=\frac{\delta}{\delta A}S_{\mathrm{CS}}={*}\big(\mathrm{d}\Sigma\wedge F\big)\sim{*}\mathrm{d}\Sigma\lrcorner F\,. \qquad (4)$$

In fact, this equation follows from generalized anti-self-duality on the cylinder $\widetilde{M}=\mathbb{R}_\tau\times M$ over M (in the A_τ=0 gauge).

The question is therefore: Which manifolds admit a global (d–4)-form? And the answer is: G-structure manifolds, i.e. manifolds with a weak special holonomy. Some of the key cases we shall encounter are listed in Table 1. For this talk I shall consider (reductive non-symmetric) coset spaces $M{=}\frac{K}{H}$ in d=6 as well as cylinders and cones over them. In all these cases, the gauge group is chosen to be K.

3 Six dimensions: nearly Kähler coset spaces

All known compact nearly Kähler 6-manifolds M^6 are nonsymmetric coset spaces K/H:

$$S^6=\frac{G_2}{\mathrm{SU}(3)}\,,\ \frac{\mathrm{Sp}(2)}{\mathrm{Sp}(1){\times}\mathrm{U}(1)}\,,\ \frac{\mathrm{SU}(3)}{\mathrm{U}(1){\times}\mathrm{U}(1)}\,,\ \mathrm{SU}(2){\times}\mathrm{SU}(2)\,. \qquad (5)$$

The coset structure $H\lhd K$ implies the decomposition

$$\mathrm{Lie}(K)\equiv\mathfrak{k}=\mathfrak{h}\oplus\mathfrak{m}\,,\quad \mathfrak{h}\equiv\mathrm{Lie}(H)\,,\quad [\mathfrak{h},\mathfrak{m}]\subset\mathfrak{m}\,. \qquad (6)$$

Interestingly, the reflection automorphism of symmetric spaces gets generalized to a so called tri-symmetry automorphism $S:K\to K$ with $S^3=\mathrm{id}$ implying $s:\mathfrak{k}\to\mathfrak{k}$ with

$$s|_{\mathfrak{h}}=\mathbb{1}\quad\text{and}\quad s|_{\mathfrak{m}}=-\tfrac12+\tfrac{\sqrt3}{2}\,J=\exp\{\tfrac{2\pi}{3}J\}\,, \qquad (7)$$

effecting a $\frac{2\pi}{3}$ rotation on TM^6. I pick a Lie-algebra basis

$$\{I_{a=1,\dots,6}\,,\,I_{i=7,\dots,\dim G}\}\,,\quad [I_a,I_b]=f^i_{ab}I_i+f^c_{ab}I_c\,, \qquad (8)$$

involving the structure constants $f^{\bullet}_{ab}$. The Cartan-Killing form then reads

$$\langle\cdot,\cdot\rangle_{\mathfrak{k}}=-\mathrm{tr}_{\mathfrak{k}}(\mathrm{ad}(\cdot)\circ\mathrm{ad}(\cdot))=3\,\langle\cdot,\cdot\rangle_{\mathfrak{h}}=3\,\langle\cdot,\cdot\rangle_{\mathfrak{m}}=\mathbb{1}\,. \qquad (9)$$

Expanding all structures in a basis of canonical one-forms e^a framing $T^*(G/H)$,

$$g=\delta_{ab}\,e^ae^b\,,\qquad \omega=\tfrac12 J_{ab}\,e^a\wedge e^b\,,\qquad \Omega=-\tfrac{1}{\sqrt3}(f+\mathrm{i}Jf)_{abc}\,e^a\wedge e^b\wedge e^c\,, \qquad (10)$$

we see that the almost complex structure (J_{ab}) and the structure constants f_{abc} rule everything.

Nearly Kähler 6-manifolds are special in that the torsion term in (2) *vanishes by itself*! What is more, this property is actually *equivalent* to the generalized anti-self-duality condition (1):

$$\begin{aligned}{*}F=-\omega\wedge F\quad&\Leftrightarrow\quad 0=\mathrm{d}\omega\wedge F\sim\mathrm{Im}\Omega\wedge F\\ &\Leftrightarrow\quad \text{DUY equations}\,,\end{aligned} \qquad (11)$$

where the Donaldson-Uhlenbeck-Yau (DUY) equations [1] state that

$$F^{2,0}=F^{0,2}=0\qquad\text{and}\qquad \omega\lrcorner F=0\,. \qquad (12)$$

Another interpretation of this anti-self-duality condition is that is projects F to the 8-dimensional eigenspace of the endomorphism ${*}(\omega\wedge\cdot)$ with eigenvalue -1, which contains the part of $F^{1,1}$ orthogonal to ω. The equations (11) imply also $\mathrm{Re}\Omega\wedge F=0$ and the (torsion-free) Yang-Mills equations $D{*}F=0$. Clearly, they seperately extremize both S_{YM} and S_{CS} in (3), but of course yield only BPS-type classical solutions. In components the above relations take the form

$$\tfrac12\epsilon_{abcdef}F_{ef}=-J_{[ab}F_{cd]}\qquad\Leftrightarrow\qquad 0=f_{abc}F_{bc} \qquad (13)$$

$$\Rightarrow\quad \omega_{ab}F_{ab}=0\,,\quad (Jf)_{abc}F_{bc}=0\,,\quad D_aF_{ab}=0\,. \qquad (14)$$

I notice that each Chern-Simons flow $\dot A_a\sim f_{abc}F_{bc}$ on M^6 ends in an instanton.

Let me look for K-equivariant connections A on M^6. If I restrict their value to $\mathfrak{h}$, the answer is

[1] also known as 'hermitian Yang-Mills equations'

unique: the only 'H-instanton' is the so called canonical connection

$$A^{\text{can}} = e^i\, I_i \qquad \Rightarrow \qquad F^{\text{can}} = -\tfrac12 f^i_{ab}\, e^a \wedge e^b\, I_i \ , \quad (15)$$

where $e^i = e^i_a e^a$. Generalizing to 'K-instantons', I extend to the ansatz

$$A = e^i\, I_i + e^a\, \Phi_{ab}\, I_b \quad \text{with} \quad \big(\Phi_{ab}\big) =: \Phi = \phi_1 \mathbb{1} + \phi_2 J \ , \quad (16)$$

with one complex parameter $\phi = \phi_1 + \mathrm{i}\phi_2$. This is in fact general for G_2 invariance on S^6. Its curvature is readily computed to

$$\begin{aligned} F_{ab} &= F^{1,1}_{ab} + F^{2,0\oplus 0,2}_{ab} \\ &= (|\Phi|^2 - 1)\, f^i_{ab}\, I_i \ + \ [(\bar\Phi^2 - \Phi)\, f]_{abc}\, I_c \end{aligned} \quad (17)$$

and displays the tri-symmetry under $\Phi \to \exp\{\frac{2\pi}{3} J\}\Phi$. The solutions to the BPS conditions (11) are

$$\bar\Phi^2 = \Phi \qquad \Rightarrow \qquad \Phi = 0 \quad \text{or} \quad \Phi = \exp\{\tfrac{2\pi k}{3} J\} \quad (18)$$

for $k = 0, 1, 2$, which yields three flat K-instanton connections besides the canonical curved one,

$$A^{(k)} = e^i\, I_i + e^a\, (s^k I)_a \qquad \text{and} \qquad A^{\text{can}} = e^i\, I_i \ . \quad (19)$$

4 Seven dimensions: cylinder over nearly Kähler

Let me step up one dimension and consider 7-manifolds M^7 with weak G_2 holonomy associated with a G_2-structure three-form ψ. Here, the 7 generalized anti-self-duality equations project F onto the -1 eigenspace of $*(\psi \wedge \cdot)$, which is 14-dimensional and isomorphic to the Lie algebra of G_2,

$$*F = -\psi \wedge F \quad \Leftrightarrow \quad *\psi \wedge F = 0 \quad \Leftrightarrow \quad \psi \lrcorner F = 0 \ , \quad (20)$$

providing an alternative form of the condition. In components, it reads

$$\tfrac12 \epsilon_{abcdefg} F_{fg} = -\psi_{[abc} F_{de]} \qquad \Leftrightarrow \qquad 0 = \psi_{abc} F_{bc} \ . \quad (21)$$

For the parallel and nearly parallel G_2 cases, the previous accident (11) recurs,

$$\mathrm{d}\psi \sim *\psi \qquad \Rightarrow \qquad \mathrm{d}\psi \wedge F = 0 \qquad \Rightarrow \qquad D{*}F = 0 \ , \quad (22)$$

and the torsion decouples. Note that on a general weak G_2-manifold there are two different flows ($\sigma \in \mathbb{R}$),

$$\frac{\mathrm{d}A(\sigma)}{\mathrm{d}\sigma} = *\mathrm{d}\psi \lrcorner F(\sigma) \quad \text{and} \quad \frac{\mathrm{d}A(\sigma)}{\mathrm{d}\sigma} = \psi \lrcorner F(\sigma) \ , \quad (23)$$

which coincide in the nearly parallel case. The second flow ends in an instanton on M^7.

In this talk I focus on cylinders $M^7 = \mathbb{R}_\tau \times \frac{K}{H}$ over nearly Kähler cosets, with a metric $g = (\mathrm{d}\tau)^2 + \delta_{ab}\, e^a e^b$. Note that these do not carry a parallel or nearly parallel G_2 structure but admit a conformally parallel one. I will study the Yang-Mills equation with a torsion given by

$$*\mathcal{H} = \tfrac13 \kappa\, \mathrm{d}\omega \wedge \mathrm{d}\tau \qquad \Leftrightarrow \qquad T_{abc} = \kappa\, f_{abc} \quad (24)$$

with a real parameter κ. We shall see that for special values of κ my torsionful Yang-Mills equation

$$D{*}F + \tfrac13 \kappa\, \mathrm{d}\omega \wedge \mathrm{d}\tau \wedge F \ = \ 0 \quad (25)$$

descends from an anti-self-duality condition (20).

Taking the $A_0{=}0$ gauge and borrowing the ansatz (16) from the nearly Kähler base, I write

$$\begin{aligned} &A_a = e^i_a I_i + [\Phi(\tau)\, I]_a \qquad \Rightarrow \qquad F_{0a} = [\dot\Phi\, I]_a \\ &\text{and} \quad F_{ab} = (|\Phi|^2 - 1)\, f^i_{ab}\, I_i \ + \ [(\bar\Phi^2 - \Phi)\, f]_{abc}\, I_c \end{aligned} \quad (26)$$

which depends on a *function* $\phi(\tau) = \phi_1(\tau) + \mathrm{i}\phi_2(\tau) \in \mathbb{C}$. Sticking this into (25) and computing for a while, one arrives at

$$\ddot\phi \ = \ (\kappa - 1)\phi \ - \ (\kappa + 3)\bar\phi^2 \ + \ 4\bar\phi\phi^2 \ =: \ \tfrac13 \frac{\partial V}{\partial\bar\phi} \ . \quad (27)$$

Nicely enough, I have obtained a ϕ^4 model with an action

$$\begin{aligned} &S[\phi] \ \sim \int_{\mathbb{R}} \mathrm{d}\tau\, \{3|\dot\phi|^2 + V(\phi)\} \qquad \text{for} \\ &V(\phi) = (3 - \kappa) + 3(\kappa - 1)|\phi|^2 - (3 + \kappa)(\phi^3 + \bar\phi^3) + 6|\phi|^4 \end{aligned} \quad (28)$$

devoid of rotational symmetry (for $\kappa \neq -3$) but enjoying tri-symmetry in the complex plane. It leads me to a mechanical analog problem of a Newtonian particle on $\mathbb{C}$ in a potential $-V$. I obtain the same action by plugging (26) directly into (3) with $\mathrm{d}\Sigma = *\mathcal{H}$ from (24).

For the case of $\frac{K}{H} = S^6 = \frac{G_2}{\mathrm{SU}(3)}$, equation (27) produces in fact *all* G-equivariant Yang-Mills connections on $\mathbb{R}_\tau \times \frac{K}{H}$. On $\frac{\mathrm{Sp}(2)}{\mathrm{Sp}(1)\times\mathrm{U}(1)}$ and $\frac{\mathrm{SU}(3)}{\mathrm{U}(1)\times\mathrm{U}(1)}$, however, the most general G-equivariant connections involves two respective three complex functions of τ. The corresponding Newtonian dynamics on $\mathbb{C}^2$ respective $\mathbb{C}^3$ is of similar type but constrained by the conservation of Noether charges related to relative phase rotations of the complex functions [3].

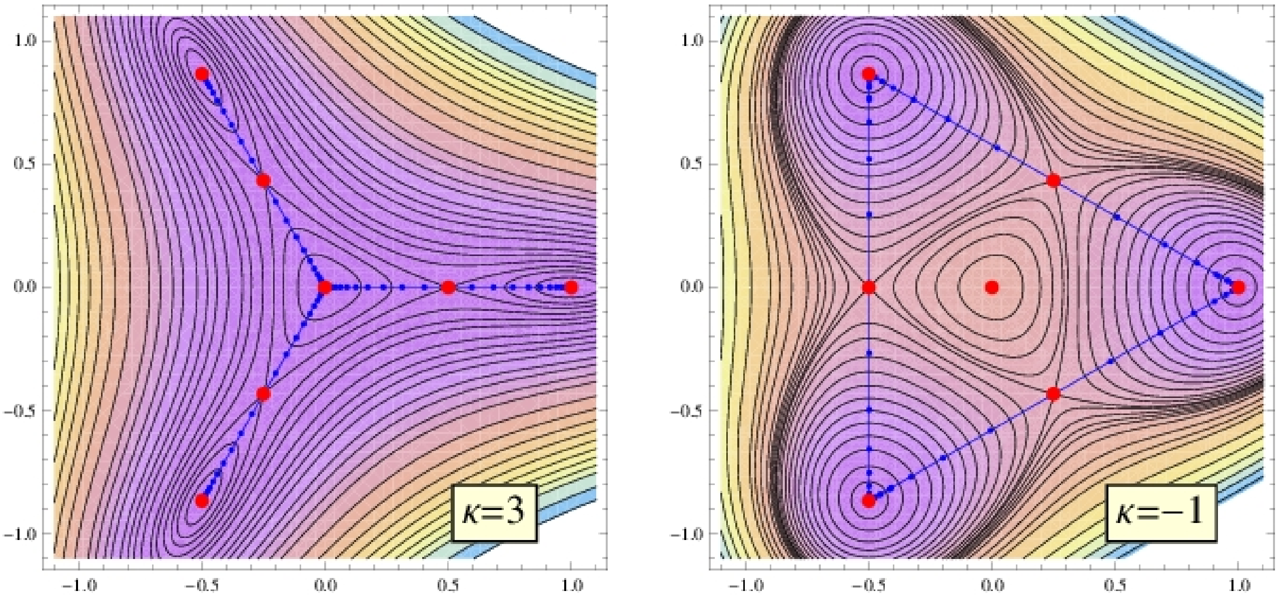


Figure 1: Contour plots of the potential and straight kink trajectories.

5 Seven dimensions: solutions

Finite-action solutions require Newtonian trajectories between zero-potential critical points $\hat{\phi}$. With two exotic exceptions, $\mathrm{d}V(\hat{\phi}) = 0 = V(\hat{\phi})$ yields precisely the BPS configurations on $\frac{K}{H}$:

- $\hat{\phi} = \mathrm{e}^{2\pi\mathrm{i}k/3}$ with $V(\hat{\phi}) = 0$ for all values of κ and $k = 0, 1, 2$
- $\hat{\phi} = 0$ with $V(\hat{\phi}) = 3{-}\kappa$ vanishing only at $\kappa{=}3$

Kink solutions will interpolate between two different critical points, while bounces will return to the critical starting point. Thus for generic κ values one may have kinks of 'transversal' type, connecting two third roots of unity, as well as bounces. For $\kappa{=}3$ 'radial' kinks, reaching such a root from the origin, may occur as well. Numerical analysis reveals the domains of existence in κ:

κ interval	$(-\infty,-3]$	$(-3,+3)$	$+3$	$(+3,+5)$
types of trajectory	radial bounce	transversal kink	radial kink	radial bounce

Magic happens at three special values of κ: At $\kappa{=}{-3}$ rotational symmetry emerges; this is a degenerate situation. At $\kappa{=}{-1}$ and at $\kappa{=}{+3}$, displayed in Figure 1 above, the kink trajectories are straight lines, indicating integrability. Indeed, behind each of these two cases lurks a first-order flow equation, which originates from anti-self-duality and hence a particular G_2-structure ψ.

Let me first discuss $\kappa{=}{+3}$. For this value I find that

$$3\ddot{\phi} = \frac{\partial V}{\partial \bar{\phi}} \quad \Leftarrow \quad \sqrt{2}\dot{\phi} = \pm\frac{\partial W}{\partial \bar{\phi}} \qquad \text{with} \quad W = \tfrac{1}{3}(\phi^3{+}\bar{\phi}^3) - |\phi|^2 \ , \tag{29}$$

which is a gradient flow with a real superpotential W, as

$$V \;=\; 6\left|\frac{\partial W}{\partial \phi}\right|^2 \qquad \text{for} \quad \kappa = +3 \ . \tag{30}$$

It admits the obvious analytic radial kink solution,

$$\phi(\tau) \;=\; \mathrm{e}^{\frac{2\pi\mathrm{i}k}{3}}\left(\tfrac{1}{2} \pm \tfrac{1}{2}\tanh\tfrac{\tau}{2\sqrt{3}}\right) \ . \tag{31}$$

What is the interpretation of this gradient flow in terms of the original Yang-Mills theory? Demanding that the torsion in (24) comes from a G_2-structure, $*\mathcal{H} = \mathrm{d}\psi$, I am led to

$$\psi = \tfrac{\kappa}{3}\omega\wedge\mathrm{d}\tau + \alpha\,\mathrm{Im}\Omega \quad \Rightarrow \quad \mathrm{d}\psi \sim \kappa\,\mathrm{Im}\Omega\wedge\mathrm{d}\tau \sim \psi\wedge\mathrm{d}\tau \tag{32}$$

where α is undetermined. This is a conformally parallel G_2-structure, and (20) quantizes the coefficients to $\alpha{=}1$ and $\kappa{=}3$, fixing (with $\mathrm{e}^\tau = r$)

$$\psi = \omega\wedge\mathrm{d}\tau + \mathrm{Im}\Omega = \tfrac{1}{r^3}\left(r^2\omega\wedge\mathrm{d}r + r^3\mathrm{Im}\Omega\right) = \tfrac{1}{r^3}\psi_{\text{cone}} \ , \tag{33}$$

where I displayed the conformal relation to the parallel G_2-structure on the cone over $\frac{K}{H}$.

Alternatively, with this G_2-structure the 7 anti-self-duality equations (20) turn into

$$\omega\lrcorner F \sim J_{ab}\,F_{ab} = 0 \quad \text{and} \quad \dot{A} \sim \mathrm{d}\omega\lrcorner F \sim e^a f_{abc} F_{bc} \ . \tag{34}$$

With the ansatz (26), the first relation is automatic, and the second one indeed reduces to (29). As a consistency check, one may verify that

$$\int_{\frac{K}{H}} \mathrm{tr}\,\{\omega\wedge F\wedge F\} \;\propto\; W(\phi)+\tfrac{1}{3}\ . \tag{35}$$

I now come to the other instance of straight trajectories, $\kappa{=}-1$. For this value I find that

$$3\ddot{\phi} = \tfrac{\partial V}{\partial\bar\phi} \quad \Leftarrow \quad \sqrt{2}\dot\phi = \pm\mathrm{i}\,\tfrac{\partial H}{\partial\bar\phi} \qquad \text{with} \quad H \;=\; \tfrac13(\phi^3{+}\bar\phi^3)-|\phi|^2\ , \tag{36}$$

which is a hamiltonian flow (note the imaginary multiplier!), running along the level curves of the function H, that is identical to W. It has the obvious analytic transverse kink solution,

$$\phi(\tau) \;=\; -\tfrac12 \pm \tfrac{\sqrt3}{2}\,\mathrm{i}\,(\tanh\tfrac{\tau}{2}) \tag{37}$$

and its images under the tri-symmetry action.

Have I discovered another hidden G_2-structure here? Let me try the other obvious choice,

$$\widetilde\psi = \tfrac{\tilde\kappa}{3}\omega{\wedge}\mathrm{d}\tau{+}\tilde\alpha\,\mathrm{Re}\Omega \quad\Rightarrow\quad \mathrm{d}\widetilde\psi \sim \tilde\kappa\,\mathrm{Im}\Omega{\wedge}\mathrm{d}\tau{+}2\tilde\alpha\,\omega{\wedge}\omega\ , \tag{38}$$

with coefficients $\tilde\kappa$ and $\tilde\alpha$ to be determined. It has not appeared in my table in Section 2, but obeys $\mathrm{d}{*}\widetilde\psi=0$, which is known as a *cocalibrated* G_2-structure. But can it produce the proper torsion,

$$\mathrm{d}\widetilde\psi{\wedge}F \sim \big(\tilde\kappa\,\mathrm{Im}\Omega{\wedge}\mathrm{d}\tau{+}2\tilde\alpha\,\omega{\wedge}\omega\big){\wedge}F \overset{!}{=} -\mathrm{Im}\Omega{\wedge}\mathrm{d}\tau{\wedge}F\ ? \tag{39}$$

Employing the anti-self-duality with respect to $\widetilde\psi$,

$$*\widetilde\psi\wedge F=0 \quad\Rightarrow\quad \omega\wedge\omega\wedge F = 2\,\mathrm{Im}\Omega\wedge\mathrm{d}\tau\wedge F\ , \tag{40}$$

it works out, adjusting the coefficients to $\tilde\kappa{=}3$ and $\tilde\alpha{=}-1$. Hence, the cocalibrated G_2-structure

$$\widetilde\psi \;=\; \omega\wedge\mathrm{d}\tau \;-\; \mathrm{Re}\Omega \tag{41}$$

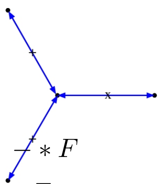
is responsible for the hamiltonian flow.

To see this directly, I import (41) into (20) and get

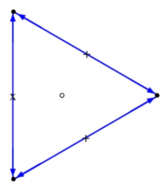
$$J_{ab}\,F_{ab} \;=\; 0 \qquad\text{and}\qquad \dot A_a \;\sim\; [J\,f]_{abc}\,F_{bc}\ . \tag{42}$$

Again, the ansatz (26) fulfils the first relation, and the second one nicely turns into (36).

6 Partial summary



Let me schematically sum up the construction.



$$\begin{array}{lcr} \psi\wedge F \;=\; *F & \text{on}\ \ \mathbb{R}\times\frac{K}{H} & \widetilde\psi\wedge F \;=\; -*F \\ \dot A_a \;\sim\; f_{abc}\,F_{bc} & & \dot A_a \;\sim\; [Jf]_{abc}\,F_{bc} \\ \downarrow & \text{ansatz}\quad A = e^i I_i + e^a[\Phi\,I]_a & \downarrow \\ \sqrt2\dot\phi = \pm\frac{\partial W}{\partial\bar\phi} & & \sqrt2\dot\phi = \pm\mathrm{i}\,\frac{\partial H}{\partial\bar\phi} \\ \downarrow & W \;=\; \frac13(\phi^3+\bar\phi^3)-|\phi|^2 \;=\; H & \downarrow \end{array}$$

$$F(\tau) \;=\; \mathrm{d}\tau{\wedge}e^a\,[\dot\Phi\,I]_a \;+\; \tfrac12 e^a{\wedge}e^b\big\{(|\Phi|^2{-}1)\,f^i_{ab}\,I_i \;+\; [(\bar\Phi^2{-}\Phi)\,f]_{abc}\,I_c\big\}$$

are G_2-instantons for Yang-Mills with torsion $\quad D*F+(*\mathcal{H})\wedge F \;=\; 0$

from $\quad S[A] \;=\; \int_{\mathbb{R}\times\frac{K}{H}}\mathrm{tr}\big\{F\wedge *F \;+\; \tfrac13\kappa\,\omega\wedge\mathrm{d}\tau\wedge F\wedge F\big\}\quad$ with $\kappa=+3$ or -1

and obey gradient or hamiltonian flow equations for $\quad \int_{\frac{K}{H}}\mathrm{tr}\,\{\omega\wedge F\wedge F\} \;\propto\; W(\phi)+\tfrac13\ .$

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ИНСТАНТОНЫ И ПОТОКИ ЧЕРНА-САЙМОНСА В ШЕСТИ И СЕМИ РАЗМЕРНОСТЯХ

Существование K-инстантонов на цилиндре $M^7 = \mathbb{R}_\tau \times \sigma \frac{K}{H}$ по однородному почти Кэлерову G-многообразию $\frac{K}{H}$ требует конформно параллельной или когрaдуируемой G_2-структуры на M^7. Обобщенная антиавтодуальность на M^7 предполагает поток Черна-Саймонса на $\frac{K}{H}$, который проходит между инстантонами на смежном классе. Для K-эквивариантных связностей, торсионные уравнения Янга-Миллса сводятся к уравнениям движения ньютоновской частицы в четвертичном потенциале на $\mathbb{C}$. Мы получаем решения для общих значений кручения. Когда последний соответствует конформно параллельной или когрaдуируемой G_2-структуре на M^7, динамика следует из градиента или гамильтонова потока, соответственно, и мы сталкиваемся с инстантонами Янга-Миллса.

Ключевые слова: *инстантоны, поток Черна-Саймонса, специальная геометрия, G-структуры, почти Кэлеровы многообразия.*

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