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Extended BRST renormalization

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The renormalization of general gauge theories curved space-time backgrounds is considered within the Sp(2)-covariant quantization method. It is proven that gauge invariant and diffeomorphism invariant renormalizability to all orders in the loop expansion and the extended BRST symmetry after renormalization is preserved.

Keywords: *Extended BRST symmetry, Sp(2) quantization, general covariance, renormalization.*

1 Introduction

In Quantum Field Theory Green's functions contain divergences [1, 2]. Renormalization is one of important means to construct a suitable quantum version for formulating fundamental interactions exiting in the Nature.

Renormalization of general gauge theories within the Batalin-Vilkovisky formalism [3,4] has been proved in papers [5, 6].

Renormalizations in curved space-time using the Dyson criterion [7] are under intense investigations beginning with paper [8] (see [9–13] and references therein). We are going to continue our investigation of gauge invariant renormalizability in curved space-time with the help of new concept of renormalizability [5]. In [14] it was done in the BV formalism [3,4]. We have extended these considerations to the case when a theory is defined in the presence of external backgrounds, in particular in curved space-time and proved that in this case the gauge invariant renormalizability is compatible with preserving general covariance.

In the present paper we consider the problem of gauge invariant renormalizability of general gauge theories in the Sp(2)-method [15–17] in the presence of a gravitational background field and to prove general covariance of renormalization.

The paper is organized as follows. In Section 2 the Sp(2) formalism the general gauge theories in the presence of an external gravitational field is considered. In Section 3 general covariance of renormalization in the Sp(2) method is proved. In Section 4 concluding remarks are given.

We use the condensed notations as given by DeWitt [18]. Derivatives with respect to sources and antifields are taken from the left, and those with respect to fields, from the right. Left derivatives with respect to fields are labeled by the subscript *"l"*. The Grassmann parity of any quantity A is denoted by $\epsilon(A)$.

2 General gauge theories in curved space within Sp(2) formalism

Let us consider a theory of gauge fields A^i in an external gravitational field $g_{\mu\nu}$. The classical theory is described by the action which depends on both dynamical fields and external metric,

$$S_0 = S_0(A, g)\,. \qquad (1)$$

Here and below we use the condensed notation $g \equiv g_{\mu\nu}$ for the metric, when it is an argument of some functional or function. The action (1) is assumed to be gauge invariant,

$$\begin{aligned} &S_{0,i}R^i_a = 0, \quad \delta A^i = R^i_a(A, g)\lambda^a, \\ &\lambda^a = \lambda^a(x) \quad (a = 1, 2, ..., n)\,, \end{aligned} \qquad (2)$$

as well as covariant,

$$\delta_g S_0 \;=\; \frac{\delta S_0}{\delta A^i}\delta_g A^i + \frac{\delta S_0}{\delta g_{\mu\nu}}\delta_g g_{\mu\nu} = 0\,, \qquad (3)$$

where λ^a are independent parameters of the gauge transformation, corresponding to the symmetry group of the theory. The diffeomorphism transformation of the metric in Eq. (3) has the form

$$\begin{aligned} \delta_g g_{\mu\nu} &= -g_{\mu\alpha}\partial_\nu\xi^\alpha - g_{\nu\alpha}\partial_\mu\xi^\alpha - \partial_\alpha g_{\mu\nu}\xi^\alpha \\ &= -g_{\mu\alpha}\nabla_\nu\xi^\alpha - g_{\nu\alpha}\nabla_\mu\xi^\alpha \\ &= -\nabla_\mu\xi_\nu - \nabla_\nu\xi_\mu\,. \end{aligned} \qquad (4)$$

Here ξ^α are the parameters of the coordinates transformation,

$$\xi^\alpha = \xi^\alpha(x) \quad (\alpha = 1, 2, ..., d)\,. \qquad (5)$$

The generating functional $Z(J, \Phi^*, \bar{\Phi}, g)$ of the Green functions can be constructed in the form of the functional integral

$$Z(J, \Phi^*, \bar{\Phi}, g) = \int d\Phi \exp\Big\{\frac{i}{\hbar}\Big[S_{ext}(\Phi, \Phi^*, \bar{\Phi}, g) + J_A\Phi^A\Big]\Big\}. \qquad (6)$$

Here Φ^A represents the full set of fields of the complete configuration space of the theory under consideration and $\Phi^*_{Aa}, \bar{\Phi}_A$ are antifields. Finally, $S_{ext}(\Phi, \Phi^*, \bar{\Phi}_A, g)$ is the quantum action constructed with the help of the solution $S = S(\Phi, \Phi^*, \bar{\Phi}_A, g)$ to the master equations

$$\frac{1}{2}(S,S)^a + V^a S = i\hbar\Delta^a S,$$
$$S(\Phi, \Phi^*, \bar{\Phi}, g)|_{\Phi^*=\bar{\Phi}=\hbar=0} = S_0(A, g). \tag{7}$$

Here on the space of fields Φ^A and antifields Φ^*_{Aa} the extended antibrackets are defined

$$(F,G)^a \equiv \frac{\delta F}{\delta\Phi^A}\,\frac{\delta G}{\delta\Phi^*_{Aa}} - -(F \leftrightarrow G)\;(-1)^{(\varepsilon(F)+1)(\varepsilon(G)+1)}. \tag{8}$$

In particular the extended antibrackets (8) obey the Jacobi identities

$$((F,G)^{\{a}, H)^{b\}}(-1)^{(\varepsilon(F)+1)(\varepsilon(H)+1)} + +\text{cycl.perm.}(F,G,H) \equiv 0, \tag{9}$$

where curly brackets denote symmetrization with respect to the indices a, b of the $Sp(2)$ group:

$$A^{\{a}B^{b\}} \equiv A^a B^b + B^b A^a.$$

In addition the operators V^a, Δ^a are introduced

$$V^a = \varepsilon^{ab}\;\Phi^*_{Ab}\,\frac{\delta}{\delta\bar{\Phi}_A}, \quad \Delta^a = (-1)^{\varepsilon_A}\frac{\delta_l}{\delta\Phi^A}\frac{\delta}{\delta\Phi^*_{Aa}}, \tag{10}$$

where ε^{ab} is the antisymmetric tensor for raising and lowering $Sp(2)$-indices

$$\varepsilon^{ab} = -\varepsilon^{ba}, \quad \varepsilon^{12} = 1 \quad \varepsilon_{ab} = -\varepsilon^{ab}.$$

One can find the algebra of the operators (10)

$$\Delta^{\{a}\Delta^{b\}} = 0, \quad \Delta^{\{a}V^{b\}} + V^{\{a}\Delta^{b\}} = 0, \quad V^{\{a}V^{b\}} = 0.$$

The action of the operators (10) on a product of functionals F and G gives

$$\Delta^a(F\cdot G) = (\Delta^a F)\cdot G + F\cdot(\Delta^a G)(-1)^{\varepsilon(F)} + +(F,\;G)^a(-1)^{\varepsilon(F)},$$

$$V^{\{a}(F,G)^{b\}} = (V^{\{a}F,G)^{b\}} - (-1)^{\varepsilon(F)}(F, V^{\{a}G)^{b\}}.$$

Note that S_{ext} satisfies the master equations

$$\frac{1}{2}(S_{ext}, S_{ext})^a + V^a S_{ext} = i\hbar\Delta^a S_{ext}. \tag{11}$$

From gauge invariance of initial action (2) in usual manner one can derive the BRST symmetry and the Ward identities for generating functionals $Z = Z(\Phi, \Phi^*, \bar{\Phi}, g), W = W(\Phi, \Phi^*, \bar{\Phi}, g)$ and ,$\Gamma = \Gamma(\Phi, \Phi^*, \bar{\Phi}, g)$ in the following form

$$\widehat{\omega}^a Z(J, \Phi^*, \bar{\Phi}) = 0, \tag{12}$$

where

$$\widehat{\omega}^a = \left(J_A\frac{\delta}{\delta\Phi^*_{Aa}} - \varepsilon^{ab}\Phi^*_{Ab}\frac{\delta}{\delta\bar{\Phi}_A}\right), \quad \widehat{\omega}^{\{a}\widehat{\omega}^{b\}} = 0, \tag{13}$$

$$\widehat{\omega}^a W(J, \Phi^*, \bar{\Phi}) = 0, \tag{14}$$

and

$$\frac{1}{2}(\Gamma,\Gamma)^a + V^a\Gamma = 0. \tag{15}$$

In what follows we assume the general covariance of $S = S(\Phi, \Phi^*, \bar{\Phi}, g)$,

$$\delta_g S = \frac{\delta S}{\delta\Phi^A}\delta_g\Phi^A + \delta_g\Phi^*_{Aa}\frac{\delta S}{\delta\Phi^*_{Aa}} + \frac{\delta S}{\delta g_{\mu\nu}}\delta_g g_{\mu\nu} = 0. \tag{16}$$

Let us choose the gauge fixing functional $F = F(\Phi, g)$ in a covariant form

$$\delta_g F = 0\,, \tag{17}$$

then the quantum action $S_{ext} = S_{ext}(\Phi, \Phi^*, \bar{\Phi}, g)$ obeys the general covariance too

$$\delta_g S_{ext} = 0\,. \tag{18}$$

From the Eq. (18) and the assumption that the term with the sources J_A in (6) is covariant

$$\delta_g(J_A\Phi^A) = (\delta_g J_A)\Phi^A + J_A(\delta_g\Phi^A) = 0\,, \tag{19}$$

it follows the general covariance of $Z = Z(J, \Phi^*, \bar{\Phi}, g)$. Indeed,

$$\delta_g Z = \frac{i}{\hbar}\int d\Phi\Big[\delta_g\Phi^*_{Aa}\frac{\delta S_{ext}}{\delta\Phi^*_{Aa}} + \frac{\delta S_{ext}}{\delta g_{\mu\nu}}\delta_g g_{\mu\nu} + +\delta_g\bar{\Phi}_A\frac{\delta S_{ext}(}{\delta\bar{\Phi}_A} + (\delta_g J_A)\Phi^A\Big] \times \exp\left\{\frac{i}{\hbar}\Big[S_{ext} + J_A\Phi^A\Big]\right\}. \tag{20}$$

Making change of integration variables in the functional integral, (20),

$$\Phi^A \quad\to\quad \Phi^A + \delta_g\Phi^A\,, \tag{21}$$

we arrive at the relation

$$\delta_g Z = \frac{i}{\hbar}\int d\Phi\Big[\frac{\delta S_{ext}}{\delta\Phi^A}\delta_g\Phi^A + \delta_g\Phi^*_A\frac{\delta S_{ext}}{\delta\Phi^*_A} + +\delta_g\bar{\Phi}_A\frac{\delta S_{ext}}{\delta\bar{\Phi}_A} + \frac{\delta S_{ext}}{\delta g_{\mu\nu}}\delta_g g_{\mu\nu} + +(\delta_g J_A)\Phi^A + J_A(\delta_g\Phi^A)\Big] \times \exp\left\{\frac{i}{\hbar}\Big[S_{ext} + J_A\Phi^A\Big]\right\} = = \frac{i}{\hbar}\int d\Phi\Big[\delta_g S_{ext} + \delta_g(J_A\Phi^A)\Big] \times \exp\left\{\frac{i}{\hbar}\Big[S_{ext} + J_A\Phi^A\Big]\right\} \;=\; 0\,. \tag{22}$$

From (22) it follows that the generating functional of connected Green functions $W(J, \Phi^*, \bar{\Phi}, g))$

$$W(J, \Phi^*, \bar{\Phi}, g) = \frac{i}{\hbar}\ln Z(J, \Phi^*, \bar{\Phi}, g) \tag{23}$$

obeys the property of the general covariance as well

$$\delta_g W(J,\Phi^*,\bar{\Phi},g)=0\,. \qquad (24)$$

Consider now the generating functional of vertex functions $\Gamma=\Gamma(\Phi,\Phi^*,\bar{\Phi},g)$

$$\Gamma(\Phi,\Phi^*,\bar{\Phi},g)=W(J,\Phi^*,\bar{\Phi},g)-J_A\Phi^A\,, \qquad (25)$$

where

$$\Phi^A=\frac{\delta W(J,\Phi^*,\bar{\Phi},g)}{\delta J_A},\quad J_A=-\frac{\delta\Gamma(\Phi,\Phi^*,\bar{\Phi},g)}{\delta\Phi^A}. \qquad (26)$$

From definition of Φ^A (26) and the general covariance of $W(J,\Phi^*,\bar{\Phi},g)$ we can conclude the general covariance of $J_A\Phi^A$. Therefore,

$$\delta_g\Gamma(\Phi,\Phi^*,\bar{\Phi},g)=\delta_g W(J,\Phi^*,\bar{\Phi},g)=0, \qquad (27)$$

the generating functional of vertex functions obyes the property of the general covariance too. So, in this Section it is proved that if an external gravitational background $g_{\mu\nu}$ does not destroy the gauge invariance of an initial action $S_0=S_0(A,g)$. then the generating functional of Green functions can be constructed with the help of solution to the Sp(2)-master equations in an usual way. Moreover, if we assume the general covariance of the initial action then we prove the general covariance of non-renormalized generating functional of Green functions as well as both the generating functional of connected Green functions and of vertex functions.

3 Covariant renormalization in curved space-time

Up to now we consider non-renormalized generating functionals of Green functions. We are going to prove the general covariance for renormalized generating functionals. For this end, let us first consider the one-loop approximation for $\Gamma=\Gamma(\Phi,\Phi^*,\bar{\Phi},g)$,

$$\Gamma=S+\hbar\big[\Gamma^{(1)}_{div}+\Gamma^{(1)}_{fin}\big]+O(\hbar^2)\,, \qquad (28)$$

where $\bar{\Gamma}^{(1)}_{div}$ and $\bar{\Gamma}^{(1)}_{fin}$ denote the divergent and finite parts of the one-loop approximation for Γ. The divergent local term $\Gamma^{(1)}_{div}$ gives the first counterpart in one-loop renormalized action S_{1R}

$$S\to S_{1R}=S-\hbar\Gamma^{(1)}_{div}. \qquad (29)$$

From (18) and (27) it follows that in one-loop approximation we have

$$\delta_g\big[\Gamma^{(1)}_{div}+\Gamma^{(1)}_{fin}\big]=0 \qquad (30)$$

and therefore $\Gamma^{(1)}_{div}$ and $\Gamma^{(1)}_{fin}$ obey the general covariance independently

$$\delta_g\Gamma^{(1)}_{div}=0\,,\qquad \delta_g\Gamma^{(1)}_{fin}=0\,. \qquad (31)$$

In its turn the one-loop renormalized action S_{1R} is covariant

$$\delta_g S_{1R}=0\,. \qquad (32)$$

Constructing the generating functional of one-loop renormalized Green functions $Z_1(J,\Phi^*,\bar{\Phi},g)$, with the action $S_{1R}=S_{1R}(\Phi,\Phi^*,\bar{\Phi},g)$, and repeating arguments given above, we arrive at the relation

$$\delta_g Z_1=0\,,\qquad \delta_g W_1=0\,,\qquad \delta_g\Gamma_1=0\,. \qquad (33)$$

The generating functional of vertex functions $\Gamma_1=\Gamma_1(\Phi,\Phi^*,\bar{\Phi},g)$ which is finite in one-loop approximation

$$\Gamma_1=S+\hbar\Gamma^{(1)}_{fin}+\hbar^2\big[\Gamma^{(2)}_{1,div}+\Gamma^{(2)}_{1,fin}\big]+O(\hbar^3)\,, \qquad (34)$$

contains the divergent part $\Gamma^{(2)}_{1,div}$ and defines renormalization of the action S in the two-loop approximation

$$S\to S_{2R}=S_{1R}-\hbar^2\Gamma^{(2)}_{1,div}\,. \qquad (35)$$

Starting from (31), (32) and (33) we derive

$$\delta_g\Gamma^{(2)}_{1,div}=0\,,\qquad \delta_g\Gamma^{(2)}_{1,fin}=0\,, \qquad (36)$$

that means general covariance of the divergent and finite parts of Γ_1 in two-loop approximation. Therefore the two-loop renormalized action $S_{2R}=S_{2R}(\Phi,\Phi^*,\bar{\Phi},g)$ is covariant

$$\delta_g S_{2R}=0. \qquad (37)$$

Applying the induction method we can repeat the procedure to an arbitrary order of the loop expansion. In this way we prove that the full renormalized action, $S_R=S_R(\Phi,\Phi^*,\bar{\Phi},g)$,

$$S_R\,=\,S-\sum_{n=1}^{\infty}\hbar^n\Gamma^{(n)}_{n-1,div}\,, \qquad (38)$$

which is local in each finite order in $\hbar$, obeys the general covariance

$$\delta_g S_R\,=\,0\,; \qquad (39)$$

and the renormalized generating functional of vertex functions, $\Gamma_R=\Gamma_R(\Phi,\Phi^*,\bar{\Phi},g))$,

$$\Gamma_R\,=\,S+\sum_{n=1}^{\infty}\hbar^n\Gamma^{(n)}_{n-1,fin}\,, \qquad (40)$$

which is finite in each finite order in $\hbar$, is covariant

$$\delta_g\Gamma_R\,=\,0\,. \qquad (41)$$

Therefore, taking into account results of Section 4 we can state that in presence of an external gravitational field the gauge invariant renormalizability can be arrived with preserving general covariance of functional Γ (41).

4 Conclusions

We have considered the general scheme of gauge-invariant and covariant renormalization of the quantum gauge theories of matter fields in flat and curved space-time. Using the Sp(2) formalism we have proved that in the theory which admits gauge invariant and diffeomorphism invariant regularization, these two symmetries hold in the counterterms to all orders of the loops expansion together with extended BRST symmetry. Indeed let us define the renormalized extended BRST operators $\hat{s}^a$

$$\hat{s}^a \bullet = (\Gamma_R, \bullet)^a + V^a \bullet .$$

Then we find that $\hat{s}^a$ satisfy the extended BRST algebra

$$\hat{s}^{\{a}\hat{s}^{b\}} = 0$$

due to the equations for Γ_R and identities for extended antibrackets and operators V^a existing in the Sp(2) formalism [15–17]. Defining the extended BRST transformations in the form

$$\delta_B \Phi^A = \hat{s}^a \Phi^A \mu_a$$

where μ_a form of the Sp(2) doublet of constant Grassmann parameters we find invariance of renormalized effective action Γ_R under these transformations on the hypersurface $\Phi^*_{Aa} = 0$

$$\delta_B \Gamma_R \big|_{\Phi^*=0} = 0.$$

Note once more that to obtain these results we have used the gauge invariant renormalizability of general gauge theories in the Sp(2) formalism without assuming the use of regularization for which acting by Δ^a on a local functional gives zero [17].

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РАСШИРЕННАЯ БРСТ ПЕРЕНОРМИРУЕМОСТЬ

Перенормируемость калибровочных теорий общего вида на искривленном пространстве-времени изучается в рамках Sp(2)-ковариантного метода квантования. Доказывается, что калибровочная инвариантность, общая ковариантность и расширенная БРСТ симметрия сохраняются во всех порядках разложения по петлям.

Ключевые слова: *Расширенная БРСТ симметрия, Sp(2)квантование, общая ковариантность, перенормировка.*

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