UDC 530.1; 539.1

Review of BRST approach to higher spin field theory

V. Krykhtin

Department of Theoretical Physics, Tomsk State Pedagogical University, Tomsk, 634061, Russia.

E-mail: krykhtin@tspu.edu.ru

We review the method which allows to construct Lagrangians for different type of fields: massive and massless, bosonic and fermionic with different index symmetry.

Keywords: *higher spin fields, gauge theory.*

1 Introduction

The method reviewed here allows to construct Lagrangians for different kind of field: in Minkowski and AdS spacetime both for bosonic and fermionic fields (massive and massless) with any index symmetry, see e.g. [1–10], in Einstein space for massive and massless spin 2 and 3/2 fields [11, 12] and for antisymmetric bosonic fields in arbitraty curved spacetime [13].

The plan of the review is as follows. First we consider basic steps of the Lagrangian construction method using the example of bosonic massive spin-s field. Next we comment on some peculiar features of Lagrangian construction for other kind of fields: the fields corresponding to arbitrary k-row Young tableau, the fields in AdS space and the fermionic fields. Finally we consider generalization of the method to the case of spin-2 and spin-3/2 in Einstein manifold.

2 Basic steps of the Lagrangian construction

Let us review the basic steps of the Lagrangian construction using example of the massive bosonic spin-s field in Minkowski spacetime.

As is known the totally symmetric tensor field $\varphi_{\mu_1\ldots\mu_s}(x)$, describing irreducible spin-s massive representation of the Pioncare group must satisfy the following constraints

$$(\partial^2 - m^2)\varphi_{\mu_1\ldots\mu_s} = 0 \tag{1}$$

$$\partial^{\mu_1}\varphi_{\mu_1\ldots\mu_s} = 0 \qquad \eta^{\mu_1\mu_2}\varphi_{\mu_1\ldots\mu_s} = 0. \tag{2}$$

Our purpose is to construct Lagrangian which reproduce these equations of motion.

The first step of the Lagrangian construction method is to rewrite these equations in the form of operator constraints

For this purpose we introduce Fock space generated by creation and annihilation operators a^+_μ, a_μ satisfying the commutation relations

$$[a_\mu, a^+_\nu] = \eta_{\mu\nu} \qquad \eta_{\mu\nu} = (-,+,\ldots,+)$$

Then we define operators

$$l_0 = \partial^2 - m^2, \qquad l_1 = ia^\mu\partial_\mu, \qquad l_2 = \tfrac{1}{2}a^\mu a_\mu$$

These operators act on states in the Fock space

$$|\varphi\rangle = \sum_{s=0}^{\infty}|\varphi_s\rangle \qquad |\varphi_s\rangle = \varphi_{\mu_1\cdots\mu_s}(x)a^{\mu_1+}\cdots a^{\mu_s+}|0\rangle$$

Each component $\varphi_{\mu_1\cdots\mu_s}(x)$ will satisfy the equations of motion (1), (2) if the following constraints on the states are fulfilled

$$l_0|\varphi\rangle = 0, \qquad l_1|\varphi\rangle = 0, \qquad l_2|\varphi\rangle = 0.$$

Thus the equations defining the irreducible representation are treated as the operators of first class constraints in some auxiliary Fock space.

Note that the operators l_0, l_1, l_2 commute each other.

To construct real Lagrangian we must construct Hermitian BRST operator. Since opertors l_1 and l_2 are not Hermitian

$$(l_1)^+ \neq l_1 \qquad (l_2)^+ \neq l_2$$

we can't construct such operator on the base of operators l_0, l_1, l_2. To construct Hermitian BRST operatror the underlying set of operators must be invariant under Hermitian conjugation.

The second step is to add more operators in order to provide such invariance.

Thus we add two more operators

$$l_1^+ = ia^{\mu+}\partial_\mu \qquad l_2^+ = \tfrac{1}{2}a^{\mu+}a^+_\mu$$

As a result, the set of operators l_0, l_1, l_2, l_1^+, l_2^+ is invariant under Hermitian conjugation.

Algebra of the operators l_0, l_1, l_2, l_1^+, l_2^+ is open in terms of commutators of these operators. To use the BRST construction in the simplest (minimal) form the underlying algebra must be closed.

Third step is to get such an algebra. For this purpose we add to the above set of operators, all operators generated by the commutators of l_0, l_1, l_2, l_1^+, l_2^+. Doing such a way we obtain two new operators

$$g_m = m^2 \qquad \text{and} \qquad g_0 = a_\mu^+ a^\mu + d/2.$$

Thus we have set of operators

$$l_0 \quad l_1 \quad l_2 \quad l_1^+ \quad l_2^+ \quad g_0 \quad g_m$$

which is invariant under Hermitian conjugation and form an algebra.

Let us note that operators l_0, l_1, l_2 are constraints in the ket-vector space

$$l_0|\varphi\rangle = l_1|\varphi\rangle = l_2|\varphi\rangle = 0,$$

operators l_0, l_1^+, l_2^+ are constraints in the bra-vector space

$$\langle\varphi|l_0 = \langle\varphi|l_1^+ = \langle\varphi|l_2^+ = 0$$

Operators g_m and g_0 are obtained from commutators

$$[l_1^+, l_1] = l_0 + g_m \qquad [l_2, l_2^+] = g_0$$

and they are not constraints neither in the bra-vector space nor in the ket-vector space

Now if we construct BRST operator we find that one of the equation on the physical state $|\varphi_s\rangle$ will have the form

$$g_m|\varphi_s\rangle = 0 \quad \Rightarrow \quad |\varphi_s\rangle = 0$$

Thus we will not reproduce the proper equations of motion. This happens due to the presence of the operators which are not constraints g_m and g_0.

Forth step. We enlarge the representation space of the operator algebra by introducing additional (new) creation and annihilation operators and enlarge expressions for the operators

$$o_i \longrightarrow O_i = o_i + o_i',$$
$$o_i \in \{l_0, l_1, l_1^+, l_2, l_2^+, g_0, g_m\}$$

The enlarged operators must satisfy two conditions:

1. They must form an algebra $[O_i, O_j] \sim O_k$;
2. The operators which can't be regarded as constraints must be zero or contain linearly an arbitrary parameter.

For our case the enlarged expressions for the opareators can be taken in the form

$$\begin{aligned}
L_0 &= \partial^2 - m^2 \\
L_1 &= ia^\mu\partial_\mu + mb_1 \\
L_1^+ &= ia^{+\mu}\partial_\mu + mb_1^+ \\
L_2 &= \tfrac{1}{2}a_\mu a^\mu + \tfrac{1}{2}b_1^2 + (b_2^+ b_2 + h)b_2 \\
L_2^+ &= \tfrac{1}{2}a_\mu^+ a^{+\mu} + \tfrac{1}{2}b_1^{+2} + b_2^+ \\
G_0 &= a_\mu^+ a^\mu + \tfrac{d+1}{2} + b_1^+ b_1 + 2b_2^+ b_2 + h \\
G_m &= 0 \\
& \quad [b_1, b_1^+] = [b_2, b_2^+] = 1
\end{aligned}$$

It is easy to see that *the operators L_2 and L_2^+ are not Hermitian conjugate to each other* if we use the usual rules for Hermitian conjugation of the additional creation and annihilation operators.

Fifth step. We change the definition of scalar product of vectors in the new representation as follows

$$\langle\Phi_1|\Phi_2\rangle_{new} = \langle\Phi_1|K|\Phi_2\rangle,$$

with some operator K. This operator K can be found in the form

$$K = \sum_{n=0}^{\infty} |n\rangle \frac{C(n,h)}{n!}\langle n|, \qquad |n\rangle = (b_2^+)^n|0\rangle,$$
$$C(n,h) = h(h+1)(h+2)\dots(h+n-1),$$
$$C(0,h) = 1.$$

Now we are ready to construct BRST operator.

Sixth step is construction of the BRST operator

$$\begin{aligned}
\tilde{Q} \;=\; & \eta_0 L_0 + \eta_1^+ L_1 + \eta_1 L_1^+ + \eta_2^+ L_2 + \eta_2 L_2^+ + \eta_g G_0 \\
& + \eta_1^+\eta_1\mathcal{P}_0 - \eta_2^+\eta_2\mathcal{P}_G \\
& + (\eta_G\eta_1^+ - \eta_2^+\eta_1)\mathcal{P}_1 + (\eta_1\eta_G - \eta_1^+\eta_2)\mathcal{P}_1^+ \\
& + 2\eta_G\eta_2^+\mathcal{P}_2 + 2\eta_2\eta_G\mathcal{P}_2^+
\end{aligned}$$

The ghost operators satisfy the usual commutation relations

$$\{\eta_0, \mathcal{P}_0\} = \{\eta_G, \mathcal{P}_G\} = \{\eta_1, \mathcal{P}_1^+\} =$$
$$= \{\eta_1^+, \mathcal{P}_1\} = \{\eta_2, \mathcal{P}_2^+\} = \{\eta_2^+, \mathcal{P}_2\} = 1$$

and act on the vacuum state as follows

$$\mathcal{P}_0|0\rangle = \mathcal{P}_G|0\rangle = \eta_1|0\rangle = \mathcal{P}_1|0\rangle = \eta_2|0\rangle = \mathcal{P}_2|0\rangle = 0.$$

The introduced operators act in the enlarged space of state vectors depending on $a^{+\mu}$, b_1^+, b_2^+ and on the ghost operators η_0, η_1^+, $\mathcal{P}_1^+$, η_2^+, $\mathcal{P}_2^+$, *but one assumes that the state vectors must be independent of the ghost η_G corresponding to the operator G_0.* The general structure of such state is

$$\begin{aligned}
|\chi\rangle \;=\; & \sum_{k_i} (b_1^+)^{k_1}(b_2^+)^{k_2}(\eta_0)^{k_3}(\eta_1^+)^{k_4}(\mathcal{P}_1^+)^{k_5} \times \\
& \times (\eta_2^+)^{k_6}(\mathcal{P}_2^+)^{k_7} a^{+\mu_1}\cdots a^{+\mu_{k_0}} \chi^{k_1\cdots k_7}_{\mu_1\cdots\mu_{k_0}}(x)|0\rangle.
\end{aligned}$$

The sum is taken over k_0, k_1, k_2, running from 0 to infinity and over k_3, k_4, k_5, k_6, k_7 running from 0 to 1. Besides for the 'physical' states we must leave in the sum only those terms which have ghost number is zero.

The last step is Lagrangian construction.

We extract the dependence of the BRST operator on the ghosts η_G, $\mathcal{P}_G$

$$\begin{aligned}
\tilde{Q} &= Q + \eta_G(\sigma + h) - \eta_2^+\eta_2\mathcal{P}_G, \\
& \quad Q^2 = \eta_2^+\eta_2(\sigma + h), \qquad [Q, \sigma] = 0, \\
\sigma &= a^{+\mu}a_\mu^+ + b_1^+ b_1 + 2b_2^+ b_2 + \eta_1^+\mathcal{P}_1 + \mathcal{P}_1^+\eta_1 \\
& \quad + 2\eta_2^+\mathcal{P}_2 + 2\mathcal{P}_2^+\eta_2 + \tfrac{d-5}{2}
\end{aligned}$$

After this, the equation on the 'physical' states in the BRST approach $\tilde{Q}|\chi\rangle = 0$ yields two equations

$$Q|\chi\rangle = 0,$$
$$(\sigma + h)|\chi\rangle = 0.$$

The last equation is the eigenvalue equation for the operator σ with the corresponding eigenvalues $-h$

$$-h \;=\; s + \tfrac{d-5}{2}, \qquad s = 0, 1, 2, \dots\,.$$

Let us denote the eigenvectors of the operator σ corresponding to the eigenvalues $s + \frac{d-5}{2}$ as $|\chi_s\rangle$

$$\sigma|\chi_s\rangle \;=\; \left(s + \tfrac{d-5}{2}\right)|\chi_s\rangle \qquad |\chi\rangle = \sum_{s=0}^{\infty} |\chi_s\rangle$$

and

$$Q_s = Q|_{-h\to s+\frac{d-5}{2}} \qquad\qquad K_s = K|_{-h\to s+\frac{d-5}{2}}$$

After this we will have equation of motion for spin-s

$$Q_s|\chi_s\rangle = 0 \qquad\qquad Q_s^2|\chi_s\rangle \equiv 0$$

and this equation of motion can be obtained from Lagrangian

$$\mathcal{L}_s \;=\; \int d\eta_0\, \langle\chi_s|K_s Q_s|\chi_s\rangle$$

The integral is taken over Grassmann odd variable η_0.

The equations of motion and Lagrangian are invariant ander the gauge transformations

$$\delta|\chi_s\rangle = Q_s|\Lambda_s\rangle, \qquad Q_s^2|\Lambda_s\rangle \equiv 0, \qquad gh(|\Lambda_s\rangle) = -1,$$
$$\delta|\Lambda_s\rangle = Q_s|\Omega_s\rangle, \qquad Q_s^2|\Omega_s\rangle \equiv 0, \qquad gh(|\Omega_s\rangle) = -2.$$

Thus the Lagrangian is constructed.

Let us repeat the basic steps of the Lagrangian construction procedure. At the starting point Lagrangian is unknown, but we know the equations of motion. Schematically the procedure looks like as follows

Equations of motion are known ⇒
⇒ we rewrite the EoM in the form of operator constraints
⇒ we add Hermitian conjugated constraints
⇒ we add operators to form an algebra
⇒ we construct enlarged expressions for the operators
⇒ we change scalar product
⇒ we construct BRST charge
⇒ *we find Lagrangian*

3 Fields corresponding to an arbitrary Young tableau

Let us consider the Lagrangian construction for the fields with index symmetry corresponding to Young tableau with 2 rows ($s_1 \geq s_2$)

$$\Phi_{\mu_1\cdots\mu_{s_1},\,\nu_1\cdots\nu_{s_2}}(x) \longleftrightarrow$$

$$\longleftrightarrow \begin{array}{|c|c|c|c|c|c|}\hline \mu_1 & \mu_2 & \cdots & \cdots & \cdots & \mu_{s_1} \\ \hline \nu_1 & \nu_2 & \cdots & \nu_{s_2} \\ \cline{1-4}\end{array}$$

The tensor field is symmetric with respect to permutation of each type of the indices $\Phi_{\mu_1\cdots\mu_{s_1},\,\nu_1\cdots\nu_{s_2}}(x) = \Phi_{(\mu_1\cdots\mu_{s_1}),\,(\nu_1\cdots\nu_{s_2})}(x)$ and in addition must satisfy the following equations

$$(\partial^2 - m^2)\Phi_{\mu_1\cdots\mu_{s_1},\,\nu_1\cdots\nu_{s_2}}(x) = 0,$$
$$\partial^{\mu_1}\Phi_{\mu_1\cdots\mu_{s_1},\,\nu_1\cdots\nu_{s_2}}(x) = \partial^{\nu_1}\Phi_{\mu_1\cdots\mu_{s_1},\,\nu_1\cdots\nu_{s_2}}(x) = 0,$$
$$\eta^{\mu_1\mu_2}\Phi_{\mu_1\mu_2\cdots\mu_{s_1},\,\nu_1\cdots\nu_{s_2}} = \eta^{\nu_1\nu_2}\Phi_{\mu_1\cdots\mu_{s_1},\,\nu_1\nu_2\cdots\nu_{s_2}} = = \eta^{\mu_1\nu_2}\Phi_{\mu_1\cdots\mu_{s_1},\,\nu_1\cdots\nu_{s_2}} = 0,$$
$$\Phi_{(\mu_1\cdots\mu_{s_1},\,\nu_1)\cdots\nu_{s_2}}(x) = 0.$$

Then we define Fock space generated by creation and annihilation operators

$$[a_i^{\mu}, a_j^{+\nu}] = \eta^{\mu\nu}\delta_{ij},$$
$$\eta^{\mu\nu} = diag(-,+,+,\cdots,+) \qquad i,j = 1,2.$$

The number of pairs of creation and annihilation operators one should introduce is determined by the number of rows in the Young tableau corresponding to the symmetry of the tensor field. Thus we introduce two pairs of such operators. An arbitrary state vector in this Fock space has the form

$$|\Phi\rangle \;=\; \sum_{s_1=0}^{\infty}\sum_{s_2=0}^{\infty} \Phi_{\mu_1\cdots\mu_{s_1},\,\nu_1\cdots\nu_{s_2}}(x) \;\times$$
$$\times\, a_1^{+\mu_1}\cdots a_1^{+\mu_{s_1}} a_2^{+\nu_1}\cdots a_2^{+\nu_{s_2}}|0\rangle.$$

To get the equations on the coefficient functions we introduce the following operators

$$l_0 = \partial^{\mu}p_{\mu} - m^2, \qquad l_i = i a_i^{\mu}\partial_{\mu},$$
$$l_{ij} = \frac{1}{2}a_i^{\mu}a_{j\mu} \qquad g_{12} = a_1^{+\mu}a_{2\mu}$$

One can check that the EoM are equivalent to

$$l_0|\Phi\rangle = 0\,, \quad l_i|\Phi\rangle = 0\,, \quad l_{ij}|\Phi\rangle = 0\,, \quad g_{12}|\Phi\rangle = 0$$

Now we can generalize this construction to the fields corresponding to k-row Young tableau. For this purpose one should introduce Fock space generated by k pairs of creation and annihilation operators, where $i,j = 1,2,\dots,k$, and then introduce operators l_0, l_i, l_{ij}, g_{ij} but now with $i,j = 1,2,\dots,k$. Operator g_{12} is generalized to operator $g_{ij} = a_i^{+\mu}a_{j\mu}$ where $i > j$.

After this the Lagrangian construction can be carried out as usual.

4 Fields in AdS

The difference of the Lagrangian construction in AdS space is that the algebra generated by the constraints is nonlinear, but it has a special structure

$$[l_i, l_j] = f_{ij}^k l_k + f_{ij}^{km} l_k l_m, \qquad f_{ij}^{km} \sim R$$

where f_{ij}^k, f_{ij}^{km} are constants. The constants f_{ij}^{km} are proportional to the scalar curvature and disappear in the flat limit. This has two consequences.

1) The algebra of the enlaged operators is changed in comparison with the algebra of the initial operators

$$[L_i, L_j] \ = \ f_{ij}^k L_k - (f_{ij}^{km} + f_{ij}^{mk}) l'_m L_k + f_{ij}^{km} L_k L_m,$$

with the additional parts satisfying the algebra additional parts

$$[l'_i, l'_j] \ = \ f_{ij}^k l'_k - f_{ij}^{km} l'_m l'_k.$$

2) Due to the algebra is nonlinear the BRST-BFV operator is defined unambiguously.

There exist different possibilities to order operators in the right hand sides of the commutators. All possible ways to order the operators can be described by some arbitrary parameters ξ_i [6, 7]. As a result the BRST operator is defined unambiguously. The arbitrariness in the BRST operator stipulated by the parameters ξ_i is resulted in arbitrariness of introducing the auxiliary fields in the Lagrangians and hence does not affect the dynamics of the basic field.

5 Fermionic fields

The Lagrangian construction for the fermionic higher spin theories have specific difference compared to the bosonic ones and demands some comments.

As before the equations of motion are reproduced from

$$Q_s|\Psi_s\rangle = 0, \qquad Q_s^2|\Psi_s\rangle \equiv 0,$$
$$\delta|\Psi_s\rangle = Q_s|\Lambda_s\rangle$$

Specific features is that in the fermionic theory we must obtain Lagrangian which is linear in derivatives. But if we try to construct Lagrangian similar to the bosonic case

$$\mathcal{L} \sim \langle\Psi|Q|\Psi\rangle$$

we obtain Lagrangian which has second order in derivatives. *To overcome this problem one first partially fixes the gauge and partially solves some field equations.* Then the obtained equations are Lagrangian ones and thus we can derive the correct Lagrangian.

6 Fields in Einstein space

Usually the Lagrangian construction in the BRST approach is carried out for fields of all spins simultaneously. The equations of motion and gauge transformations

$$Q|\Psi_s\rangle = 0 \qquad \delta|\Psi_s\rangle = Q|\Lambda_s\rangle$$

with the BRST operator Q being the same for all spins. Nilpotency of the BRST operator provides us the gauge transformations and fields $|\Psi_s\rangle$ and $|\Psi_s\rangle + Q|\Lambda_s\rangle$ are both physical. Since we consider all spins simultaneously

$$Q^2|\Lambda_s\rangle = 0 \quad \Rightarrow \quad Q^2 = 0$$

. But if we want to construct Lagrangian for the field with a given value s of spin, then it is sufficient to require a weaker condition.

The BRST operator for given spin Q_s may be not nilpotent in operator sense but will be nilpotent only on the specific Fock vector parameter $|\Lambda_s\rangle$ corresponding to a given spin s

$$Q_s^2|\Lambda_s\rangle = 0 \qquad \text{but} \qquad Q_s^2 \neq 0$$

on states of general form. *Just this point allows us to construct Lagrangians for spin-2 and spin-3/2 fields in Einstein background [11, 12].*

7 Summary

- We have considered the basic principles of gauge invariant Lagrangian construction for higher spin fields.
- This method can be applied to
  - any bosonic and fermionic fields in Minkowski and (A)dS spaces with any index symmetry
  - antisymmetric bosonic fields in arbitrary curved space-time
  - spin-2 and spin-3/2 fields in Einstein space
- No off-shell constraints on the fields and gauge parameters are imposed.
- The obtained Lagrangians possess a reducible gauge invariance.

Acknowledgement

This research has been supported in parts by the grants Kadry, project No 14.B37.21.1301, RFBR-Ukraine grant, project No 11-02-90445, RFBR grant project No. 12-02-00121-a and the grant for LRSS, project No 224.2012.2 .

References

[1] Pashnev A. and Tsulaia M. Mod. Phys. Lett. A **13** (1998) 1853 [hep-th/9803207].

[2] Buchbinder I. L. and Krykhtin V. A., Nucl. Phys. B **727** (2005) 537 [hep-th/0505092].

[3] Buchbinder I. L., Krykhtin V. A. and Pashnev A. Nucl. Phys. B **711** (2005) 367 [hep-th/0410215].

[4] Buchbinder I. L., Krykhtin V. A., Ryskina L. L. and Takata H. Phys. Lett. B **641** (2006) 386 [hep-th/0603212].

[5] Buchbinder I. L., Pashnev A. and Tsulaia M. Phys. Lett. B **523** (2001) 338 [hep-th/0109067].

[6] Buchbinder I. L., Krykhtin V. A. and Lavrov P. M. Nucl. Phys. B **762** (2007) 344 [hep-th/0608005].

[7] Buchbinder I. L., Krykhtin V. A. and Reshetnyak A. A. Nucl. Phys. B **787** (2007) 211 [hep-th/0703049].

[8] Burdik C., Pashnev A. and Tsulaia M. Mod. Phys. Lett. A **16** (2001) 731 [hep-th/0101201].

[9] Buchbinder I. L., Krykhtin V. A. and Takata H. Phys. Lett. B **656** (2007) 253 [arXiv:0707.2181 [hep-th]].

[10] Buchbinder I. L., Krykhtin V. A. and Ryskina L. L. Nucl. Phys. B **819** (2009) 453 [arXiv:0902.1471 [hep-th]].

[11] Buchbinder I. L., Krykhtin V. A. and Lavrov P. M. Phys. Lett. B **685** (2010) 208 [arXiv:0912.0611 [hep-th]].

[12] Buchbinder I. L. and Krykhtin V. A., Mod. Phys. Lett. A **25** (2010) 1667 [arXiv:1003.0185 [hep-th]].

[13] Buchbinder I. L., Krykhtin V. A. and Ryskina L. L. Mod. Phys. Lett. A **24** (2009) 401 [arXiv:0810.3467 [hep-th]].

Received 01.10.2012

В. А. Крыхтин

БРСТ ПОДХОД К ТЕОРИИ ПОЛЕЙ ВЫСШИХ СПИНОВ. ОБЗОР

Даётся обзор БРСТ подхода к построению лагранжианов для полей высших спинов. На примере массивных бозонных полей высших спинов в пространстве Минковского показаны все этапы построения лагранжинов с помощью данного подхода. Рассказано также о специфике построения лагранжианов для фермионных полей высших спинов, полей высших спинов в пространстве постоянной кривизны и полей спина 2 и 3/2 в пространстве Эйнштейна.

Ключевые слова: *поля высших спинов, калибровочная симметрия.*

Крыхтин В. А., кандидат физико-математических наук.
Томский государственный педагогический университет.
Ул. Киевская, 60, Томск, Россия, 634061.
E-mail: krykhtin@tspu.edu.ru