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MINISUPERSPACE APPROACH OF GENERALIZED GRAVITATIONAL MODELS

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1 Introduction

It is well known that recently there has been found strong evidence for an accelerate expansion of our universe, apparently due to the so called dark energy. With regard to this issue, here we would like to make some considerations involving general relativistic theories of gravitation. In fact, recently alternative and geometric descriptions for the dark energy in modern cosmology have been proposed and discussed in several related issues [1,2]. Such models are higher derivative gravitational theories, thus they may contain instabilities [3] and deviation from Newton gravity [4]. However, if one takes quantum effects into account, one can get a viable theory [5]. The Palatini method has also been applied in consistent way [6,7,8] and the evaluation of the black hole entropy within these models has been investigated in [9].

To this aim, we shall consider a general relativistic theories, (see for example [10,11]), namely let us assume that our model is described by the action

$$S=\frac{1}{16\pi G}\int d^4x\sqrt{-g}\,f(R), \tag{1.1}$$

with $f(R)$ depending only on the scalar curvature.

As first example, let us consider the Lagrangian [1]

$$f(R)=\left[R-\frac{\mu^4}{R}\right], \tag{1.2}$$

where μ is a new cosmological parameter [1]. As is well known , there exist constant curvature de Sitter and AdS vacuum solutions such that

$$R_0^2=3\mu^4. \tag{1.3}$$

Another well known example, is given by the choice

$$f(R)=R+\gamma R^2-2\Lambda, \tag{1.4}$$

where the the other possible quadratic term giving by the Weyl invariant has been omitted because is vanishing for space-time we are dealing with (see, for example [10]).

As a third example, let us consider an effective Coleman-Weinberg like model

$$f(R)=R+R^2\left(\gamma+\beta\ln\left(\frac{R}{\mu^2}\right)\right), \tag{1.5}$$

where γ, β and μ are suitable constants.

2 Minisuperspace approach

Our aim in this section will be the issue of a minisuperspace Lagrangian description, in order to investigate classical and quantum aspects, like the stability and canonical quantization. For these reasons, one has to restrict to FRW isotropic and homogeneous metrics with constant spatial section. We choose a spatial flat metric, namely

$$ds^2=a^2(\eta)(-N^2(\eta)d\eta^2d^2\vec{x}), \tag{2.1}$$

where η is the conformal time, $a(\eta)$ the cosmological factor and $N(\eta)$ an arbitrary lapse function, which describes the gauge freedom associated with the reparametrization invariance of the minisuperspace gravitational model. For the above metric, the scalar curvature reads

$$R=6\left(\frac{a''}{a^3N^2}-\frac{a'N'}{a^3N^3}\right), \tag{2.2}$$

in which $'$ stands for $\frac{d}{d\eta}$.

If one plugs this expression in the Eq. (1.1), one obtains, an higher derivative Lagrangian theory. Higher derivative Lagrangian theory may be treated canonically by means the Ostrogadski method (see, for example [12] and references cited therein).

Here we have found more convenient to follow the method outlined in ref. [13]. To deal with a non standard higher derivatives Lagragian system, we make use of a Lagrangian multiplier λ and we write

$$S=\frac{V_3}{16\pi G}\int d\eta Na^4\times \\ \times\left[L_c(R)-\lambda\left(R-6\left(\frac{a''}{a^3N^2}-\frac{a'N'}{a^3N^3}\right)\right)\right]. \tag{2.3}$$

Making the variation with respect to R, one gets

$$\lambda=\frac{df(R)}{dR}. \tag{2.4}$$

Thus, substituing this value and making a standard integration by part, one arrives at the Lagrangian, which will be our starting point

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$$L(a,a',R,R') = -6\frac{a'^2}{N}\frac{df(R)}{dR} - 6\frac{a'aR'}{N}\frac{d^2f(R)}{dR^2} + \\ +Na^4\left(f(R) - R\frac{df(R)}{dR}\right). \quad (2.5)$$

It should be noted that N appears as "einbein" Lagrangian multiplier, as it should be, reflecting the parametrization invariance of the action. In fact the Lagragian is quasi-invariant with respect to the infinitesimal gauge transformation

$$\delta a = \epsilon(\tau)a'(\tau), \quad \delta R = \epsilon(\tau)R'(\tau), \quad \delta N = \frac{d}{d\tau}[\epsilon(\tau)N(\tau)]. \quad (2.6)$$

As a consequence, we have the (energy) constraint

$$\frac{\partial L}{\partial N} = 0, \quad (2.7)$$

namely

$$6\frac{a'^2}{N^2}\frac{df(R)}{dR} + 6\frac{a'aR'}{N^2}\frac{d^2f(R)}{dR^2} - \\ -a^4\left(f(R) - R\frac{df(R)}{dR}\right) = 0, \quad (2.8)$$

and we may choose, for example, the gauge $N=1$. The other Eqs. of motion are

$$\frac{d}{d\eta}\left[\frac{2a'}{N}\frac{df(R)}{dR} + \frac{aR'}{N}\frac{d^2f(R)}{dR^2}\right] = \\ = \frac{a'R'}{N}\frac{d^2f(R)}{dR^2} - \frac{2}{3}a^3\left(f(R) - R\frac{df(R)}{dR}\right), \quad (2.9)$$

$$\frac{d}{d\eta}\left[\frac{aa'}{N}\frac{d^2f(R)}{dR^2}\right] = \frac{RNa^3}{6}\frac{d^2f(R)}{dR^2} + \\ +\frac{a'^2}{N}\frac{d^2f(R)}{dR^2} + \frac{a'R'a}{N}\frac{d^3f(R)}{dR^3}. \quad (2.10)$$

The conserved quantity is the energy, computed with the standard Legendre transformation

$$E = -6\frac{a'^2}{N}\frac{df(R)}{dR} - 6\frac{a'aR'}{N}\frac{d^2f(R)}{dR^2} - \\ -Na^4\left(f(R) - R\frac{df(R)}{dR}\right). \quad (2.11)$$

E is vanishing on shell due to the Eq. of motion for the einbein N.

We shall be interested in models which admit solution with constant 4-dimensional curvature of the de Sitter type, namely

$$R = R_0, \quad a_0 = \frac{A}{\eta}, \quad A^2R_0 = 12. \quad (2.12)$$

If we plug this particular solutions in the above Eqs. of motion, we get the condition [10]

$$2f(R_0) = R_0\frac{df(R)}{dR}(R_0), \quad (2.13)$$

which may be used to find the constant curvature R_0. For the model defined by Eq. (1.2), Eq. (2.13) leads again to the condition

$$R_0^2 = 3\mu^4, \quad (2.14)$$

while for the Lagrangian (1.4), Eq. (2.13) gives

$$R_0 = 4\Lambda, \quad (2.15)$$

and for the Coleman-Weinberg like model gives

$$R_0 = \frac{1}{c_2}, \quad R_0 = 0. \quad (2.16)$$

It is easy to check that such kind of solutions are physically ones, because we have

$$E = \frac{6A^2}{\eta^4}\left[1 - \frac{2f(R_0)}{R_0\frac{df}{dR}(R_0)}\right], \quad (2.17)$$

namely they satisfy identically the energy constraint $E=0$. Thus, the condition (2.13) turns out to be a necessary and sufficient condition in order to have physical constant curvature solutions.

In order to investigate the Hamiltonian formalism, it is convenient to make the following change of variables [13]: $N \to N$, $a \to q$ and $R \to \phi$, defined by

$$\frac{df(R)}{dR} = B^2e^{2\phi} \quad (2.18)$$

$$a = qe^{-\phi}. \quad (2.19)$$

In the first, B is a suitable constant, fixed by means of

$$B^2 = \frac{df(R)}{dR}(R_0), \quad (2.20)$$

and R, as a function of the new variable ϕ, is defined implicitely $R = R(\phi)$.

For example, for the choice (1.2), one has

$$\frac{df(R)}{dR}(R) = 1 + \frac{\mu^4}{R^2} = B^2e^{2\phi}, \quad (2.21)$$

and

$$R = \frac{\mu^2}{(B^2e^{2\phi} - 1)^{1/2}}. \quad (2.22)$$

For the Lagrangian (1.4), one obtains

$$R = \frac{B^2e^{2\phi} - 1}{2\gamma}. \quad (2.23)$$

In the case of Coleman-Weinberg like model, one only has

$$B^2e^{2\phi} = 1 + 2R\left(\gamma + \beta\ln\frac{R}{\mu^2}\right) + \beta R. \quad (2.24)$$

Thus, it is not possibile to obtain explicetily R as a function of the new variable ϕ.

The de Sitter like solution corresponds to

$$\phi_0 = 0, \quad q_0 = \frac{A}{\eta}, \quad A^2R_0 = 12. \quad (2.25)$$

A direct calculation leads to Lagrangian

$$L = \frac{6}{N}[(2q - q^2)^2\phi'^2 - q'^2] - NV(\phi, q), \quad (2.26)$$

in which the potential reads

$$V(\phi,q)=-\frac{q^4}{B^2}e^{-4\phi}\left[f(R(\phi))-R\frac{df(R)}{dR}(\phi)\right]. \qquad (2.27)$$

For example, for the Lagrangian (1.2), we have (see also[1,2,3])

$$V(\phi,q)=\frac{q^4e^{-4\phi}}{B^2}[B^2e^{2\phi}-1]^{1/2}, \qquad (2.28)$$

while for the Lagrangian (1.4), one obtains

$$V(\phi,q)=\frac{q^4}{B^2}e^{-4\phi}\left[2\Lambda+\frac{(B^2e^{2\phi}-1)^2}{4\gamma}\right], \qquad (2.29)$$

The conserved energy reads

$$E=\frac{6}{N}[(2q-q^2)^2\phi'^2-q'^2]+NV(\phi,q), \qquad (2.30)$$

and it is vanishing on shell. The corresponding Hamiltonian is

$$H=N\left[\frac{1}{24(2q-q^2)}P_\phi^2-\frac{1}{24}P_q^2+V(\phi,q)\right]. \qquad (2.31)$$

The Lagrangian equation of motion, in the gauge $N=1$, read

$$q''-(1-q)\phi'^2+ +\frac{q^3}{3}\frac{e^{-4\phi}}{B^2}\left[f(R)-R\frac{df(R)}{dR}\right]=0, \qquad (2.32)$$

$$(2q-q)^2\phi''+2(q'-qq')\phi'- -\frac{q^4}{6B^2}e^{-4\phi}\left[-2f(R)+R\frac{df(R)}{dR}\right]=0. \qquad (2.33)$$

If the Lagrangian satisfies the condition (2.13), it is easy to check that the one has again the de Sitter solution (2.25).

We conclude this Section writing down the equations for the small disturbances around the de Sitter solution, namely

$$q=q_0+\delta q, \quad \phi=\delta\phi. \qquad (2.34)$$

Taking Eq. (2.13) into account again, the equations for the small disturbance around de Sitter solution turn out to be

$$\frac{d^2\delta q}{d^2\eta}-\frac{6}{\eta^2}\delta q=0, \qquad (2.35)$$

$$\left(2-\frac{A}{\eta}\right)\frac{d^2\delta\phi}{d^2\eta}-\frac{2}{\eta}\left(1-\frac{A}{\eta}\right)\frac{d\delta\phi}{d\eta}- -4R_0\frac{A^3}{\eta^3}\left[1-\frac{2f(R_0)}{R_0^2\frac{d^2f}{d^2R_0}}\right]\delta\phi=0. \qquad (2.36)$$

Some remarks are in order. First the small disturbance equations are decoupled in the conformal time. Second, the equation associated with the variable q is universal, namely it does not depend explicitely on function $f(R)$, but we remind that the constant value R_0 depends on it. Third, the general solution of the first equation is not hard to find and reads

$$\delta q=c_1\eta^3+c_2\eta^{-2}. \qquad (2.37)$$

However, the perturbed solution $q=q_0+\delta q$ must satisfy the energy constraint $E=0$ and this leads to

$$\delta E=0, \qquad (2.38)$$

namely, around the de Sitter solution

$$q_0'\delta q-q_0''\delta q=0. \qquad (2.39)$$

As a result $c_1=0$ and we have

$$q=\frac{A}{\eta}\left[1+\frac{\eta_0^2\delta q_0}{A\eta}\right]. \qquad (2.40)$$

Recall that the relation between the conformal time and the cosmological time t may be written

and $12A=R_0^2$, R_0 being the de Sitter curvature and $t=0$ corresponds to η_0.

$$\eta=\eta_0e^{\frac{t}{A}}, \qquad (2.41)$$

and $12A=R_0^2$, R_0 being the de Sitter curvature and $t=0$ corresponds to η_0.

Thus, the solution remains small with respect the de Sitter one for

$$\eta>\frac{12\eta_0^2\delta q_0}{R_0^2}, \qquad (2.42)$$

R_0 being the de Sitter curvature.

Along the same lines, one may investigate the Starobinski model [14] and its generalization including the Brans-Dicke field [15] and its brane-world generalization [16].

3 Conclusions

In this paper, we have presented a minisuperspace approach for general relativistic pure gravitational models. The inclusion of the matter can be easily taken into account. A canonical approach has been presented by means of the methods of ref. [13]. These models are, in general, instable, due to the presence, at the beginning, of higher derivative terms [3]. However, the inclusion of quantum effects may resolve the problem [5].

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CASIMIR ENERGY AND THE COSMOLOGICAL CONSTANT

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I. Introduction

One of the most fascinating and unsolved problems of the theoretical physics of our century is the cosmological constant. Einstein introduced his cosmological constant Λ_c in an attempt to generalize his original field equations. The modified field equations are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda_c g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{1}$$

where Λ_c is the cosmological constant, G is the gravitational constant and $T_{\mu\nu}$ is the energy-momentum tensor. By redefining

$$T_{\mu\nu}^{tot} \equiv T_{\mu\nu} - \frac{\Lambda_c}{8\pi G} g_{\mu\nu}, \tag{2}$$

one can regain the original form of the field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}^{tot} = 8\pi G (T_{\mu\nu} + T_{\mu\nu}^{\Lambda}), \tag{3}$$

at the prize of introducing a vacuum energy density and vacuum stress-energy tensor

$$\rho_\Lambda = \frac{\Lambda_c}{8\pi G}, \quad T_{\mu\nu}^{\Lambda} = -\rho_\Lambda g_{\mu\nu}. \tag{4}$$

Alternatively, Eq. (1) can be cast into the form,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda_{eff} g_{\mu\nu} = 0, \tag{5}$$

where we have included the contribution of the vacuum energy density in the form $T_{\mu\nu} = -\langle\rho\rangle g_{\mu\nu}$. In this case Λ_c can be considered as the bare cosmological constant

$$\Lambda_{eff} = 8\pi G \rho_{eff} = \Lambda_c + 8\pi G \langle\rho\rangle. \tag{6}$$

Experimentally, we know that the effective energy density of the universe ρ_{eff} is of the order $10^{-47}\,\mathrm{GeV}^4$. A crude estimate of the Zero Point Energy (ZPE) of some field of mass m with a cutoff at the Planck scale gives

$$E_{ZPE} = \frac{1}{2} \int_0^{\Lambda_p} \frac{d^3k}{(2\pi)^3} \sqrt{k^2 + m^2} \simeq \frac{\Lambda_p^4}{16\pi^2} \approx 10^{71}\,\mathrm{GeV}^4. \tag{7}$$

This gives a difference of about 118 orders [1]. The approach to quantization of general relativity based on the following set of equations

$$\left[\frac{2\kappa}{\sqrt{g}} G_{ijkl} \pi^{ij} \pi^{kl} - \frac{\sqrt{g}}{2\kappa} (R - 2\Lambda_c) \right] \Psi[g_{ij}] = 0 \tag{8}$$

and

$$-2\nabla_i \pi^{ij} \Psi[g_{ij}] = 0, \tag{9}$$

where R is the three-scalar curvature, Λ_c is the bare cosmological constant and $\kappa = 8\pi G$, is known as Wheeler-De Witt equation (WDW) [2]. Eqs. (8) and (9) describe the *wave function of the universe*. The WDW equation represents invariance under *time* reparametrization in an operatorial form, while Eq. (9)

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