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Landau models with both worldline and target supersymmetries: $\mathcal{N}=2$ and $\mathcal{N}=4$ examples

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We present a synopsis of superextended Landau models possessing both worldline supersymmetry and graded internal symmetry acting in the target space. The main focus is on the recently constructed model with the worldline $\mathcal{N}=4$ supersymmetry and the target $ISU(2|2)$ symmetry.

Keywords: *supersymmetry, Landau levels, superspace.*

1 Introduction

The original quantum Landau model [1] describes a charged particle moving on a plane orthogonal to a constant uniform magnetic flux. Its generalization is the spherical Landau-type model [2] which describes a charged particle on the 2-sphere $S^2 \sim SU(2)/U(1)$ in the Dirac monopole background.

Superextensions of the Landau and Haldane models deal with non-relativistic particles moving on supergroup manifolds with S^2 or its planar limit as a "body". Minimal superextensions of the S^2 Haldane model are:

1. Landau problem on the $(2+2)$-dimensional supersphere $SU(2|1)/U(1|1)$ [3,4];
2. Landau problem on the $(2+4)$-dimensional superflag $SU(2|1)/[U(1)\times U(1)]$ [4,5].

Their large S^2 radius limits yield planar super Landau models [6–9]. Most surprising feature of the super planar Landau problems is the hidden world-line $\mathcal{N}=2$ supersymmetry. Thus, the super planar Landau models simultaneously provide a class of supersymmetric quantum mechanics (SQM) models. SQM models [10] have a plenty of applications in diverse domains.

A natural approach to constructing super planar Landau models is as follows. One takes the notion of the world-line $\mathcal{N}=2$ SUSY as the primary one and reproduces the planar Landau model and its most general $\mathcal{N}=2$ supersymmetric version from a worldline superfield formalism [9,11]. Recently, this "bottom-up" approach was applied for constructing the first example of super Landau model with the worldline $\mathcal{N}=4$ supersymmetry [12]. We found the target space supersymmetry $ISU(2|2)$ as a natural generalization of the $ISU(1|1)$ symmetry of the $\mathcal{N}=2$ case. The present talk is devoted to a short account of this construction, with collecting, as a prerequisite, some salient facts about the ordinary and $\mathcal{N}=2$ supersymmetric Landau models.

Note that sigma models with the supergroup target spaces received much attention for the last years, in particular, in connection with superbranes (see, e.g., [13–15]).

2 Bosonic Landau models

The Lagrangian and Hamiltonian of the planar bosonic Landau model are given by the following expressions

$$L_b = |\dot z|^2 - i\kappa\left(\dot z\bar z - \dot{\bar z} z\right) = |\dot z|^2 + \left(A_z \dot z + A_{\bar z}\dot{\bar z}\right), \qquad (1)$$

where

$$A_z = -i\kappa\bar z, A_{\bar z} = i\kappa z, \partial_{\bar z}A_z - \partial_z A_{\bar z} = -2i\kappa\,,$$

and

$$H_b = \frac{1}{2}\left(a^\dagger a + a a^\dagger\right) = a^\dagger a + \kappa\,, \qquad (2)$$

where

$$a = i(\partial_{\bar z} + \kappa z), \quad a^\dagger = i(\partial_z - \kappa\bar z), \quad [a, a^\dagger] = 2\kappa\,.$$

The invariances of the model are the "magnetic translations" and $2D$ rotations generated by

$$P_z = -i(\partial_z + \kappa\bar z), P_{\bar z} = -i(\partial_{\bar z} - \kappa z), F_b = z\partial_z - \bar z\partial_{\bar z},$$

$$[P_z, P_{\bar z}] = 2\kappa,\ [H, P_z] = [H, P_{\bar z}] = [H, F_b] = 0.$$

The n-th Landau level (LL) wave function is defined as

$$\Psi_{(n)}(z,\bar z) = [i(\partial_z - \kappa\bar z)]^n e^{-\kappa|z|^2}\psi_{(n)}(z),$$
$$H\Psi_{(n)} = \kappa(2n+1)\Psi_{(n)}\,.$$

Each LL is infinitely degenerate due to $(P_z, P_{\bar z})$ invariance.

The S^2 generalization of the planar Landau model is defined by the following $SU(2)$ invariant Lagrangian

$$L_b = \frac{1}{(1+|z|^2)^2}|\dot z|^2 + is\frac{1}{1+|z|^2}\left(\dot z\bar z - \dot{\bar z}z\right), \qquad (3)$$

where the 2nd term is the $d=1$ Wess-Zumino term on the coset $SU(2)/U(1)\sim S^2$.

The relevant wave functions are finite-dimensional $SU(2)$ irreps, $s, s+1, s+2, \dots s+\ell$ being their "spins". The energy spectrum is determined by the formula

$$E_\ell = \ell(2s+\ell+1)+2s\,, \quad \ell = 0,1,2,\dots\,. \tag{4}$$

Each LL is finitely degenerated since wave functions are $SU(2)$ irreps. Redefining $z\to r\,z$, $H\to H\,r^2$, $sr^2=\kappa$, where r is the "inverse" radius of S^2, in the limit $r\to 0$, with κ fixed, one recovers the planar Landau model.

3 Planar super Landau models

Planar super Landau models are the large radius limits of the supersphere and superflag Landau models. In this limit, the supersphere $SU(2|1)/U(1|1)$ goes into an $(2+2)$-dim. superplane.

The Lagrangian and Hamiltonian of the superplane Landau model read:

$$\begin{aligned}
&L = L_f + L_b = |\dot z|^2 + \dot\zeta\dot{\bar\zeta} - i\kappa\left(\dot z\bar z - \dot{\bar z}z + \dot\zeta\bar\zeta + \dot{\bar\zeta}\zeta\right),\\
&H = a^\dagger a - \alpha^\dagger\alpha = \partial_{\bar\zeta}\partial_\zeta - \partial_z\partial_{\bar z}\\
&+\kappa\left(\bar z\partial_{\bar z} + \bar\zeta\partial_{\bar\zeta} - z\partial_z - \zeta\partial_\zeta\right) + \kappa^2\left(z\bar z + \zeta\bar\zeta\right).
\end{aligned}$$

The invariances are generated by $P_z, P_{\bar z}, \Pi_\zeta = \partial_\zeta + \kappa\bar\zeta, \Pi_{\bar\zeta} = \partial_{\bar\zeta} + \kappa\zeta$ and the new generators

$$\begin{aligned}
&Q = z\partial_\zeta - \bar\zeta\partial_{\bar z}\,,\; Q^\dagger = \bar z\partial_{\bar\zeta} + \zeta\partial_z\,,\\
&C = z\partial_z + \zeta\partial_\zeta - \bar z\partial_{\bar z} - \bar\zeta\partial_{\bar\zeta}\,.
\end{aligned} \tag{5}$$

They form the algebra of the supergroup $ISU(1|1)$, contraction of $SU(2|1)$:

$$\{Q,Q^\dagger\} = C\,,\; [Q,P] = i\Pi\,,\; \{Q^\dagger,\Pi\} = iP\,. \tag{6}$$

The natural $ISU(1|1)$-invariant inner product

$$<\phi|\psi> = \int d\mu\; \overline{\phi\left(z,\bar z;\zeta,\bar\zeta\right)}\,\psi\left(z,\bar z;\zeta,\bar\zeta\right),\; d\mu = d^2z d^2\zeta,$$

leads to negative norms for some component wave functions. All norms can be made positive by introducing the "metric" operator:

$$\begin{aligned}
&G = \frac{1}{\kappa}\left[\partial_\zeta\partial_{\bar\zeta} + \kappa^2\bar\zeta\zeta + \kappa\left(\zeta\partial_\zeta - \bar\zeta\partial_{\bar\zeta}\right)\right],\\
&<<\phi|\psi>> \sim \int d\mu\; \overline{(G\phi)}\,\psi\,.
\end{aligned}$$

Note that H commutes with G, so $H = H^\dagger = H^\ddagger$. However, the hermitian conjugation properties of the operators which *do not commute* with G, change. Let $\mathcal{O}$ be a symmetry generator, such that $[H,\mathcal{O}] = 0$. Then

$$\mathcal{O}^\ddagger \equiv G\mathcal{O}^\dagger G = \mathcal{O}^\dagger + G\mathcal{O}_G^\dagger\,,\; \mathcal{O}_G \equiv [G,\mathcal{O}]\,,$$

and O_G is another operator such that $[H,\mathcal{O}_G]=0$. The symmetry generators that do not commute with G thus generate, in general, additional "hidden" symmetries.

In our case G commutes with all $ISU(1|1)$ generators, except Q, $Q^\dagger$, hence

$$\begin{aligned}
&Q^\ddagger = Q^\dagger - \frac{i}{\kappa}S,\; S = i\left(\partial_z\partial_{\bar\zeta} + \kappa^2\bar z\zeta - \kappa\bar z\partial_{\bar\zeta} - \kappa\zeta\partial_z\right),\\
&S = a^\dagger\alpha\,,\; S^\ddagger = a\alpha^\ddagger\,.
\end{aligned}$$

The operators $S, S^\ddagger, H$ form $\mathcal{N}=2, d=1$ superalgebra

$$\begin{aligned}
&\{S,S^\ddagger\} = 2\kappa H\,,\quad \{S,S\} = \{S^\ddagger,S^\ddagger\} = 0\,,\\
&[H,S] = [H,S^\ddagger] = 0\,.
\end{aligned}$$

The LLL ground state is annihilated by S and $S^\ddagger$:

$$S\psi^{(0)} = S^\ddagger\psi^{(0)} = 0,$$

and so it is $\mathcal{N}=2$ SUSY singlet. Hence $\mathcal{N}=2$ SUSY is unbroken and all higher LL form its irreps.

4 $\mathcal{N}=2$ superfield formulation

One can recover the planar super Landau model from another end, just taking the manifest $\mathcal{N}=2$ worldline supersymmetry as an input [9].

The starting point is $\mathcal{N}$=2, d=1 superspace in the left-chiral basis, $(\theta,\bar\theta,\tau\equiv t+i\theta\bar\theta)$. The basic objects are $\mathcal{N}$=2, d=1 chiral bosonic and fermionic superfields $\Phi = z(\tau)+\theta\chi(\tau)$, $\Psi = \bar\zeta(\tau)+\theta h(\tau)$, with $\chi(\tau)$ and $h(\tau)$ being auxiliary fields. The superfield action yielding the superplane model action is:

$$S = -\kappa\int dt d^2\theta\left\{\Phi\bar\Phi + \Psi\bar\Psi + \rho\left[\Phi D\Psi - \bar\Phi\bar D\bar\Psi\right]\right\},$$

where $\rho = 1/(2\sqrt{\kappa}), \kappa\neq 0$. On shell, the fields h and χ are expressed as $\chi = i/\sqrt{\kappa}\,\dot\zeta\,,\quad h = i/\sqrt{\kappa}\,\dot{\bar z}$. In terms of physical fields:

$$S \Rightarrow \int dt\Big[i\kappa\left(z\dot{\bar z} - \bar z\dot z + \zeta\dot{\bar\zeta} - \dot\zeta\bar\zeta\right) + \left(\dot z\dot{\bar z} + \dot\zeta\dot{\bar\zeta}\right)\Big].$$

This approach triggered the idea [11] to construct generalized $\mathcal{N}=2$ Landau models by passing to the most general $\mathcal{N}=2$ superfield action:

$$\begin{aligned}
S_{gen} &= \int dt d^2\theta\Big[K(\Phi,\bar\Phi) + V(\Phi,\bar\Phi)\Psi\bar\Psi\\
&+ \rho\left(\Phi D\Psi - \bar\Phi\bar D\bar\Psi\right)\Big] = \int dt\mathcal{L}\,.
\end{aligned} \tag{7}$$

This action involves two independent superfield potentials, $K(\Phi,\bar\Phi)$, $V(\Phi,\bar\Phi)$. Eliminating auxiliary fields in $\Phi = z+\dots$, $\Psi = \psi+\dots$, one gets

$$\mathcal{L}_{comp} = V^{-1}\,\dot z\dot{\bar z} + i\left(\dot z K_z - \dot{\bar z}K_{\bar z}\right) + \;(\psi-\text{terms})\,. \tag{8}$$

The superpotential V defines a target metric, while K produces a background super gauge field.

5 $\mathcal{N}=4$ superfield action

$\mathcal{N}=4$ counterpart of the $\mathcal{N}=2$ Landau model action is written in *bi-harmonic* superspace (bi-HSS) [16]:

$$S_{N=4}=-\frac{i\kappa}{2}\Big(\int \mu^{-2,0}q^{(1,0)A}q^{(1,0)B}C_{AB} - i\int \mu^{0,-2}\psi^{(0,1)A}\psi^{(0,1)B}\epsilon_{AB} + \frac{1}{\sqrt{\kappa}}\int \mu^{-2,0}q^{(1,0)A}D^{1,-1}\psi^{(0,1)B}\epsilon_{AB}\Big). \tag{9}$$

It involves two bi-harmonic superfields, $q^{(1,0)A}$ and $\psi^{(0,1)B}$, which live on two different analytic subspaces of the $\mathcal{N}=4, d=1$ bi-HSS. Without entering into details, the superfields $q^{(1,0)A}$, $\psi^{(0,1)B}$ amount to the following sets of the off-shell components:

$$q^{(1,0)A}\Rightarrow (f^{iA},\ \psi^{aA}(t))\,,\ \psi^{0,1A}\Rightarrow(\chi^{aA}(t),h^{iA}(t))\,.$$

The fields (f^{iA},χ^{aA}) are physical, while (ψ^{aA},h^{iA}) are auxiliary. The off-shell component lagrangian reads:

$$L\propto (2i\dot f^{iA}f^B_i-\psi^{aA}\psi^B_a)C_{AB} + (2\dot\chi^{aA}\chi^B_a - ih^{iA}h^B_i)\epsilon_{AB} - \frac{2i}{\sqrt{\kappa}}(\dot f^{iA}h^B_i+\psi^{aA}\dot\chi^B_a)\epsilon_{AB}\,. \tag{10}$$

After eliminating auxiliary fields as $h_{iA}=-\frac{1}{\sqrt{\kappa}}\dot f_{iA}$, $\psi_{aA}=\frac{i}{\sqrt{\kappa}}C_{AB}\dot\chi^B_a$, one obtains

$$L=\kappa C_{AB}\dot f^{iA}f^B_i - i\kappa\dot\chi^{aA}\chi_{aA} + \frac{1}{2}\left(\dot f^{iA}\dot f_{iA}+iC_{AB}\dot\chi^{aA}\dot\chi^B_a\right). \tag{11}$$

The Lagrangian (11) involves the Lorentz force-type coupling to the external gauge field:

$$\mathcal{A}_{iB}\dot f^{iB}\,,\quad \mathcal{A}_{iB}=-\kappa C_B{}^D f_{iD}\,.$$

This field is self-dual:

$$\mathcal{F}_{iA\,jB}:=\partial_{iA}\mathcal{A}_{jB}-\partial_{jB}\mathcal{A}_{iA}=-2\kappa\, C_{AB}\epsilon_{ij}\,.$$

The bosonic sector of (11) corresponds to the model used in [17] to describe $U(1)$ quantum Hall effect on $\mathbb{R}^4$.

6 Symmetries of $\mathcal{N}=4$ model

The action (11) respects $\mathcal{N}=4$ supersymmetry:

$$\delta f^{iA}=-\frac{i}{\sqrt{\kappa}}\,\epsilon^{ia}C^{AB}\dot\chi_{aB},\quad \delta\chi^{aA}=-\frac{1}{\sqrt{\kappa}}\,\epsilon^{ia}\dot f^A_i\,,$$

where ϵ^{ia} are four Grassmann parameters. The corresponding conserved Noether supercharge is

$$S_{ia}=-\frac{i}{\sqrt{\kappa}}\,C^{AB}\dot\chi_{aA}\dot f_{iB}$$

After quantization it becomes the generator of $\mathcal{N}=4$ supersymmetry

Besides the worldline supersymmetry the action respects "internal" supersymmetry which is realized by differential operators in the target (4 + 4) superspace (f^{iA},χ^{aB}). Their set involves:

A. "Magnetic" supertranslations:

$$P_{iA}=-i\partial_{f^{iA}}+\kappa\, C_{AB}f^B_i,\ \Pi_{aA}=\partial_{\chi^{aA}}+\kappa\,\chi_{aA}\,,$$
$$[P_{iA},P_{jB}]=2\kappa\epsilon_{ij}C_{AB},\ \{\Pi_{aA},\Pi_{bB}\}=2\kappa\epsilon_{ab}\epsilon_{AB}\,.$$

B. Superrotations:

$$Q^{ia}=\frac{1}{2}(iC^A_B-\delta^A_B)\chi^{aB}\partial_{f^A_i}+\frac{1}{2}(iC^A_B+\delta^A_B)f^{iB}\partial_{\chi^A_a}\,,$$
$$\{Q^{ia},\bar Q_{jb}\}=\delta^a_bT^{(i}{}_{j)}-\delta^i_jT^{(a}{}_{b)}+\frac{1}{2}i\delta^i_j\delta^a_b\,Z,$$
$$\{Q^{ia},Q^{jb}\}=0\,.$$

Here, $Z=C^A_B(f^{iB}\partial_{f^{iA}}+\chi^{aB}\partial_{\chi^{aA}})$ is $U(1)$ generator, $T^i{}_k, T^a{}_b$ are generators of two automorphism $SU(2)$ groups acting on the doublet indices i and a.

The fifteen generators $Q^{ia},\bar Q^{jb},T^i{}_k,T^a{}_b,Z$ form the superalgebra $su(2|2)$, a graded version of $su(4)$. The full target space superalgebra is

$$(P_{iA},\Pi_{aA})\rtimes SU(2|2)=ISU(2|2)\,.$$

It is a natural generalization of the target $ISU(1|1)$ symmetry of the $\mathcal{N}=2$ super Landau model.

7 Quantization

We use the complex fields:

$$z=f^{11},u=f^{21},\zeta=\chi^{11},\xi=\chi^{21},\text{and c.c.},$$
$$L=|\dot z|^2+|\dot u|^2-i\kappa(\dot z\bar z-\dot{\bar z}z+\dot u\bar u-\dot{\bar u}u) + \dot\zeta\dot{\bar\zeta}+\dot\xi\dot{\bar\xi}-i\kappa(\dot\zeta\bar\zeta+\dot{\bar\zeta}\zeta+\dot\xi\bar\xi+\dot{\bar\xi}\xi).$$

The quantum Hamiltonian reads

$$H_q=a^\dagger_za_z+a^\dagger_ua_u-\alpha^\dagger_\zeta\alpha_\zeta-\alpha^\dagger_\xi\alpha_\xi\,.$$
$$a_z=i(\frac{\partial}{\partial\bar z}+\kappa z)\,,a_u=i(\frac{\partial}{\partial\bar u}+\kappa u)\,,$$
$$\alpha_\zeta=\frac{\partial}{\partial\bar\zeta}-\kappa\zeta,\ \alpha_\xi=\frac{\partial}{\partial\bar\xi}-\kappa\xi$$
$$[a_z,a^\dagger_z]=[a_u,a^\dagger_u]=2\kappa,$$
$$\{\alpha_\zeta,\alpha^\dagger_\zeta\}=\{\alpha_\xi,\alpha^\dagger_\xi\}=-2\kappa.$$

The $\mathcal{N}=4$ supercharges read:

$$S^{11}:=S_2=\frac{i}{\sqrt{\kappa}}(\alpha^\dagger_\xi a_z-a^\dagger_u\alpha_\zeta),$$
$$S^{21}:=S_1=\frac{i}{\sqrt{\kappa}}(\alpha^\dagger_\xi a_u+a^\dagger_z\alpha_\zeta),$$
$$\{S_1,S^\dagger_1\}=\{S_2,S^\dagger_2\}=-2H_q$$

(the sign will change after redefining the inner product). In the covariant notation:

$$\{S^{ia}, S^{jb}\} = 2\epsilon^{ij}\epsilon^{ab}H_q.$$

For S^{ia} and H_q there exists a Sugawara representation

$$S^{ia} = 2\sqrt{\kappa}(Q^{ia} + \bar{Q}^{ia}) - \frac{i}{\sqrt{\kappa}}P^i_A\Pi^{aA},$$
$$H = \frac{1}{2}P^{iA}P_{iA} + \frac{i}{2}C^{AB}\,\Pi^a_B\Pi_{aA} + 2\kappa iZ$$

One can also define one more $\mathcal{N}=4$ supersymmetry

$$\hat{S}^{ia} = 2i\sqrt{\kappa}(Q^{ia} - \bar{Q}^{ia}) + \frac{i}{\sqrt{\kappa}}P^i_A\Pi^a_B C^{AB},$$
$$\{\hat{S}^{ia}, \hat{S}^{jb}\} = 2\epsilon^{ij}\epsilon^{ab}H_q.$$

The closure of the two $\mathcal{N}=4$ superalgebras is the worldline $su(2|2)$:

$$\{S^{ia}, \hat{S}^{jb}\} = 8i\kappa\left(\epsilon^{ab}\hat{T}^{ij} - \epsilon^{ij}\hat{T}^{ab}\right), \tag{12}$$

$\hat{T}^{ij}$ and $\hat{T}^{ab}$ being the R-symmetry $SU(2)$ generators.

The wave functions are defined as follows:

A. The lowest Landau level (LLL), $H\Psi^0 = 0$:

$$\Psi^0 = e^{-\kappa K}\psi_0, \quad K = |z|^2 + |u|^2 + \zeta\bar{\zeta} + \xi\bar{\xi},$$
$$\psi_0(z,u,\zeta,\xi) = A^0(z,u) + \zeta B^0(z,u) + \xi C^0(z,u) + \zeta\xi D^0(z,u).$$

The LLL wave function has a four-fold degeneracy: A^0, B^0, C^0, D^0 are closed under $ISU(2|2)$. It is a singlet of $\mathcal{N}=4$ supersymmetry.

B. Next LLs, $H\Psi^{(N)} = 2\kappa N\Psi^{(N)}$, with $N > 0$.

Introducing the $SU(2)$ covariant notation $a^i \equiv (a_z\ a_u)$, ; $\alpha^a \equiv (\alpha_\zeta\ \alpha_\xi)$,, $[a^\dagger_i, a^j] = 2\kappa\delta^j_i$, $\{\alpha^\dagger_a, \alpha^b\} = 2\kappa\delta^b_a$, the wave function $\Psi^{(N)}$ can be constructed as

$$\Psi^{(N)} = a^\dagger_{(i_1}a^\dagger_{i_2}...a^\dagger_{i_N)}e^{-\kappa K}\phi^{(i_1i_2...i_N)}(z,u,\zeta,\xi)$$
$$+\alpha^\dagger_a a^\dagger_{(i_1}a^\dagger_{i_2}...a^\dagger_{i_{N-1})}e^{-\kappa K}\psi^{a(i_1i_2...i_{N-1})}(z,u,\zeta,\xi)$$
$$+(\alpha^\dagger)^2 a^\dagger_{(i_1}a^\dagger_{i_2}...a^\dagger_{i_{N-2})}e^{-\kappa K}\phi^{(i_1i_2...i_{N-2})}.(z,u,\zeta,\xi)$$

The component wave functions form irreps of the worldline $\mathcal{N}=4$. The $\mathcal{N}=4$ supermultiplet for N-th level collects the $SU(2)$ spins

$$\left(s_1 = \frac{N}{2}, \quad s_2 = \frac{N-1}{2}, s_3 = \frac{N-2}{2}\right). \tag{13}$$

The degeneracy of the N-th level is

$$4[(2s_1+1) + 2(2s_2+1) + (2s_3+1)] = 16N. \tag{14}$$

Like in other super Landau models, with the standard definition of the inner product

$$\langle\phi|\psi\rangle = \int d\mu\overline{\phi(f,\chi)}\psi(f,\chi),\ d\mu = d^4fd^4\chi$$

there are negative norms. Let $\Psi_{(l,m)}$, $l,m = 0,1$ be a wave function with one or two fermionic quanta $\alpha^\dagger_\zeta$ and $\alpha^\dagger_\xi$. Then

$$\langle\Psi_{(l,m)}|\Psi_{(l,m)}\rangle \sim (-1)^{l+m}(||D_{(l,m)}||^2 + 2\kappa||B_{(l,m)}||^2 + 2\kappa||C_{(l,m)}||^2 + 2\kappa^2||A_{(l,m)}||^2),$$
$$||f||^2 := \int dzd\bar{z}dud\bar{u}e^{-2\kappa|z|^2-2\kappa|u|^2}\overline{f(z,u)}f(z,u).$$

The states with $\Psi_{(1,0)}$ and $\Psi_{(0,1)}$ have negative norm.

To cure this, one redefines the inner product as in the $\mathcal{N}=2$ case

$$\langle\langle\psi|\phi\rangle\rangle := \langle G\psi|\phi\rangle,$$
$$G = (1-2n_\zeta)(1-2n_\xi),\ n_{\zeta,\xi} := \frac{\alpha^\dagger_{\zeta,\xi}\,\alpha_{\zeta,\xi}}{2\kappa}.$$

The metric operator G possesses the standard properties

$$[H_q, G] = 0, \quad G^2 = 1.$$

With the new definition of the hermitian conjugation, H_q becomes manifestly positive-definite:

$$H = a^\dagger_z a_z + a^\dagger_u a_u + \alpha^\ddagger_\zeta\alpha_\zeta + \alpha^\ddagger_\xi\alpha_\xi.$$

8 Generalized $\mathcal{N}=4$ models

Most general $\mathcal{N}=4$ supersymmetric action of the superfields $q^{(1,0)A}$ and $\psi^{(0,1)A}$ is

$$S_{gen} = \frac{\kappa}{2i}\Big(\int\mu^{-2,0}L^{2,0}(q^{(1,0)A},u,v) - i\int\mu^{0,-2}\psi^{(0,1)A}\psi^{(0,1)B}\epsilon_{AB} + \frac{1}{\sqrt{\kappa}}\int\mu^{-2,0}q^{(1,0)A}D^{1,-1}\psi^{(0,1)B}\epsilon_{AB}\Big),$$

where $L^{2,0}$ is an arbitrary function of its arguments. After going to components and eliminating auxiliary fields one obtains the on-shell Lagrangian

$$L = \dot{f}^{iA}\mathcal{A}_{iA} - i\kappa\dot{\chi}^{aA}\chi_{aA} + \frac{1}{2}\dot{f}^{iA}\dot{f}_{iA} + i\frac{\kappa}{2}(G^{-1})_{AB}\dot{\chi}^{aA}\dot{\chi}^B_a.$$

Here

$$\mathcal{A}_{iA}(f) = -\kappa\int dudv\,u^{-1}_i\frac{\partial L^{2,0}}{\partial f^{(1,0)A}},$$
$$G_{AB}(f) = \frac{\kappa}{2}\int dudv\,\frac{\partial^2 L^{2,0}}{\partial f^{(1,0)A}\partial f^{(1,0)B}}.$$

The external gauge field $\mathcal{A}_{iA}(f)$ by construction is selfdual on $\mathbb{R}^4$,

$$\mathcal{F}_{iBjA} := \partial_{iB}\mathcal{A}_{jA} - \partial_{jA}\mathcal{A}_{iB} = -2G_{AB}\epsilon_{ij}.$$

In the bosonic sector we obtain some generalization of the Elvang, Polchinski $U(1)$ model in $\mathbb{R}^4$, with an arbitrary self-dual external gauge field. All other terms in the Lagrangian can be brought to the same form as in the original $\mathcal{N}=4$ super Landau model. How to obtain non-trivial target metric as in the general $\mathcal{N}=2$ Landau model? This is an open question. Perhaps one should make use of some nonlinear versions of the off-shell $\mathcal{N}=4$ multiplets involved [18].

9 Outlook

Among possible physical applications of the super Landau models outlined here (including the new $\mathcal{N}=4$ model), as well as the problems for the further study, we would like to distinguish the following ones.

These models might constitute a basis of possible supersymmetric versions of the Quantum Hall Effect (QHE) in diverse dimensions. For instance, as follows from our consideration, QHE on $\mathbb{R}^4$ [17] could have a natural $\mathcal{N}=4$ superextension described by the $\mathcal{N}=4$ Landau model. Also, we expect a close relation of the models considered to integrable structures in the planar $\mathcal{N}=4, d=4$ SYM theory and string theory. Indeed, the integrable $su(2|2)$ and $su(3|2)$ spin chains play an important role in these theories [19, 20], and it would be hardly accidental that the supergroups of similar type appear as the target space symmetries in the quantum-mechanical super Landau models. In this context, it is of clear interest to construct and study curved analogs of the $\mathcal{N}=4$ Landau model based, e.g., on the supercoset $SU(3|2)/U(2|2) \sim \mathbb{C}^{(2|2)}$. Our planar model should be reproduced from such a curved system in the contraction limit $R\to\infty$. The $SU(3|2)$ system can be treated as a superextension of one of the $SU(3)/U(2)$ models studied in [21] in connection with the four-dimensional QHE.

It is interesting to generalize our consideration to higher $\mathcal{N}$ worldline supersymmetries (e.g. $\mathcal{N}=8$) and to construct alternative $\mathcal{N}=4$ super Landau models based on other known off-shell $\mathcal{N}=4, d=1$ multiplets, e.g. $(\mathbf{3},\mathbf{4},\mathbf{1})$ and $(\mathbf{2},\mathbf{4},\mathbf{2})$. The supersymmetric Landau-type models with couplings to external non-abelian gauge fields (introduced by methods of ref. [22]) also present an ambitious subject for the future investigations.

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МОДЕЛИ ЛАНДАУ С СУПЕРСИММЕТРИЯМИ НА МИРОВОЙ ЛИНИИ И В ПРОСТРАНСТВЕ ОТОБРАЖЕНИЯ: $\mathcal{N}=2$ И $\mathcal{N}=4$ ПРИМЕРЫ

Представлен синопсис суперрасширенных моделей Ландау, обладающих как суперсимметрией на мировой линии, так и градуированной внутренней симметрией, реализованной в пространстве отображения. Основное внимание уделено недавно построенной модели с мировой $\mathcal{N}=4$ суперсимметрией и $ISU(2|2)$ симметрией пространства отображения.

Ключевые слова: *суперсимметрия, модель Ландау, суперпространство.*

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