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NEAR HORIZON BLACK HOLE GEOMETRIES AND INTEGRABLE MODELS

A. Galajinsky

Laboratory of Mathematical Physics, Tomsk Polytechnic University, 634050 Tomsk, Lenin pr., 30, Russia.

E-mail: galajin@tpu.ru

Near horizon geometry of an extremal rotating black hole in $d = 2n$ dimensions is considered and superintegrable mechanics associated with this geometry is constructed.

Keywords: *black holes, conformal mechanics, integrable models.*

1 Introduction

A remarkable property of the near horizon extremal black hole in arbitrary dimension is that its isometry group involves the conformal factor $SO(2,1)$ [1]. Because Killing vectors are linked to first integrals of the geodesic equations, a massive relativistic particle propagating on such a background inherits the conformal invariance and belongs to the class of conformal mechanics models. A salient feature of this system is that, by applying a suitable canonical transformation, the radial canonical pair can be separated from angular variables and the model can be put in the conventional conformal mechanics form [2,3]. Because the variables are separated, the angular sector can be studied in its own right [4]. In particular, the Casimir element of the conformal algebra $so(2,1)$ realized in the original relativistic particle determines the Hamiltonian of a reduces mechanics.

The purpose of this work is to consider the near horizon geometry of an extremal rotating black hole in $d = 2n$ dimensional spacetime and to construct a superintegrable mechanics associated to it.

2 Near horizon geometry of the extremal maximally symmetric $d = 2n$ black hole

A generalization of the Kerr solution of the Einstein equations to the case of even–dimensional space–time was proposed in [5]. In Boyer–Lindquist–type coordinates it reads

$$ds^2 = dt^2 - \frac{U}{\Delta}dr^2 - \frac{2M}{U}\left(dt - \sum_{i=1}^{n-1} a_i \mu_i^2 d\phi_i\right)^2 - \sum_{i=1}^{n}(r^2+a_i^2)d\mu_i^2 - \sum_{i=1}^{n-1}(r^2+a_i^2)\mu_i^2 d\phi_i^2,$$

$$\Delta = \frac{1}{r}\prod_{i=1}^{n-1}(r^2+a_i^2) - 2M,$$

$$U = r\sum_{i=1}^{n}\frac{\mu_i^2}{r^2+a_i^2}\prod_{j=1}^{n-1}(r^2+a_j^2), \tag{1}$$

where the latitudinal coordinates μ_i obey the constraint

$$\sum_{i=1}^{n}\mu_i^2 = 1. \tag{2}$$

It is assumed that only $(n-1)$ independent rotation parameters are present so a_n is set to zero in (1). The range of azimuthal coordinates ϕ_i is taken to be $[0,2\pi]$, μ_i lie in the interval $[0,1]$ for $i = 1,\dots,n-1$, while $\mu_n \in [-1,1]$. The isometry group of (1) includes the time translation and $(n-1)$ rotations which altogether form $U(1)^n$.

Because in this work we are primarily concerned with the construction of superintegrable systems, in what follows we consider only the special case that all the rotation parameters are equal $a_i = a$. In this case the $U(1)^{n-1}$ subgroup in the isometry group, which corresponds to rotations, is known to enhance to $U(n-1)$.

Before implementing the near horizon limit one has to put the metric in a more convenient form

$$ds^2 = \frac{\Delta}{U}\left(dt - a\sum_{i=1}^{n-1}\mu_i^2 d\phi_i\right)^2 - \frac{U}{\Delta}dr^2 - \frac{(r^2+a^2)^{n-2}}{rU}\sum_{i=1}^{n-1}\mu_i^2\left(a dt - (r^2+a^2)d\phi_i\right)^2 - (r^2+a^2)\sum_{i=1}^{n-1}d\mu_i^2 - r^2 d\mu_n^2 + \frac{a^2(r^2+a^2)^{n-1}}{rU}\sum_{i<j}^{n-1}\mu_i^2\mu_j^2(d\phi_i - d\phi_j)^2,$$

$$\Delta = \frac{1}{r}(r^2+a^2)^{n-1} - 2M,$$

$$U = \frac{1}{r}(r^2+a^2)^{n-2}(r^2+a^2\mu_n^2),$$

$$\mu_n^2 = 1 - \sum_{i=1}^{n-1} \mu_i^2. \tag{3}$$

The extremal solution is characterized by the condition that Δ has a double zero at the horizon radius $r = r_0$.

In order to implement the near horizon limit, one redefines the coordinates

$$r \to r_0 + \epsilon r_0 r, \qquad t \to \frac{\alpha t}{\epsilon}, \tag{4}$$

$$\phi_i \to \phi_i + \frac{\beta_i t}{\epsilon},$$

and adjusts the number coefficients α and β_i in such a way that, up to a factor, the first two terms in (3) produce the AdS_2 metric in the limit $\epsilon \to 0$

$$\alpha = \frac{2(n-1)r_0}{2n-3}, \qquad \beta_i = \frac{r_0}{a}. \tag{5}$$

Sending ϵ to zero, one derives the near horizon metric

$$ds^2 = \rho_0^2 \left(r^2 dt^2 - \frac{dr^2}{r^2} \right) - 2(n-1) \sum_{i=1}^{n-1} d\mu_i^2$$

$$-d\mu_n^2 - \frac{4}{(2n-3)^2 \rho_0^2} \sum_{i=1}^{n-1} \mu_i^2 (r dt + d\phi_i)^2 +$$

$$+ \frac{2}{(n-1)(2n-3)\rho_0^2} \sum_{i<j}^{n-1} \mu_i^2 \mu_j^2 (d\phi_i - d\phi_j)^2,$$

$$\rho_0^2 = \frac{1 + (2n-3)\mu_n^2}{2n-3}, \quad \mu_n^2 = 1 - \sum_{i=1}^{n-1} \mu_i^2, \tag{6}$$

where we discarded an overall factor of r_0^2 and scaled the azimuthal angular variables as follows: $\frac{a(n-1)}{r_0}\phi_i \to \phi_i$. It is straightforward to verify that (6) is a vacuum solution of the Einstein equations.

3 Conformal mechanics in $d = 2n$ and its integrable reductions

In order to construct the Hamiltonian of a massive relativistic particle propagating on the curved background (6), one inverts the metric (6) and then solves the mass shell condition

$$g^{nm} p_n p_m = m^2, \tag{7}$$

where $p_m = (p_0, p_r, p_{\mu_i}, p_{\phi_i})$, for p_0. This gives the Hamiltonian $p_0 = H$. Then one computes the generators of the conformal group $so(2,1)$ and the Casimir element of $so(2,1)$ which determines a reduced integrable spherical mechanics

$$\tilde{H} = \frac{1}{(2n-3)(2n-2)} \sum_{i,j=1}^{n-1} ((2n-3)\rho_0^2 \delta_{ij}$$

$$-\mu_i \mu_j) p_{\mu_i} p_{\mu_j} + \sum_{i,j=1}^{n-1} \left(\frac{(2n-3)(2n-2)\rho_0^2}{4\mu_i^2} \delta_{ij} \right.$$

$$\left. - \frac{(2n-3)^2 \rho_0^2}{4} - 1 \right) p_{\phi_i} p_{\phi_j} + m^2 \rho_0^2, \tag{8}$$

where ρ_0^2 is given in (6) and m^2 is now treated as a coupling constant. By construction, it inherits $U(n-1)$ symmetry of the background. A further reduction occurs if one disregards the cyclic variables ϕ_i and sets p_{ϕ_i} to be constants in (8)

$$H' = \frac{1}{(2n-3)(2n-2)} \sum_{i,j=1}^{n-1} ((2n-3)\rho_0^2 \delta_{ij}$$

$$-\mu_i \mu_j) p_{\mu_i} p_{\mu_j} + \sum_{i=1}^{n-1} \frac{g_i^2 \rho_0^2}{\mu_i^2} + \nu \sum_{i=1}^{n-1} \mu_i^2,$$

$$\rho_0^2 = \frac{2(n-1)}{2n-3} - \sum_{i=1}^{n-1} \mu_i^2. \tag{9}$$

Here ν and g_i are coupling constants. Thus, we have constructed a new variant of integrable spherical mechanics with $2(n-1)$ phase space degrees of freedom. Further analysis shows that this model lacks only one ontegral of motion to be maximally superintegrable [6].

4 Conclusion

To summarize, in this work we have constructed a metric describing the near horizon geometry of an extremal rotating black hole in $d = 2n$ dimensions for the special case that all the rotation parameters are equal. The Hamiltonian of an integrable spherical mechanics associated with such a geometry was constructed.

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А. В. Галажинский

ГЕОМЕТРИЯ ЭКСТРЕМАЛЬНЫХ ЧЕРНЫХ ДЫР ВБЛИЗИ ГОРИЗОНТА И ИНТЕГРИРУЕМЫЕ СИСТЕМЫ

Исследуется геометрия экстремальной вращающейся черной дыры в пространстве d=2n измерений вблизи горизонта событий. Строится суперинтегрируемая система, ассоциированная с такой геометрией.

Ключевые слова: *черные дыры, конформная механика, интегрируемые системы.*

Галажинский А.В., доктор физико-математических наук.
Томский политехнический университет.
Пр. Ленина, 30, 634050 Томск, Россия.
E-mail: galajin@tpu.ru