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The field of precessing magnetic dipole

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The field of a rotating inclined magnetic dipole is studied. One first integral of motion and some particular solutions of equations of motion for a charged particle in this field are found. The effective potential energy of a charge is studied. It is shown that the effective potential energy in the corotating reference system has six stationary points, which correspond to circular motion of a charge.

Keywords: *magnetic dipole, electromagnetic field, inclined rotator, charge, equations of motion.*

1 Introduction

Magnetic field of the planets and stars can be thought of as a dipole field in good approximation. Motion of a charged particle in the field of magnetic dipole has been investigated in details since 1907. In particular, the Earth magnetic field was studied by Størmer [1,2]. Solution of equation of motion for charged particle in a dipolar magnetic field gives trapping regions for particles, with energy of limited range. These regions of planets are known as radiation belts [3,4]. The first theoretical studies on the properties of the trajectories of charged particle in a dipolar field are presented in [1,5,6].

There are also well known bodies, for which direction of magnetic moment is different from the direction of axis of rotation. In this case there is an electric field around the body induced by magnetic field. The neutron stars and pulsars are examples of such bodies. Some models of electric field which is generated in the neighbourhood of neutron stars were developed in [7]. All these models are based on suggestion that the neutron star is a conducting body. But there are also celestial bodies, which consist of non-conducting matter and their axis of rotation is inclined with respect to the magnetic field axis. Magnetic field of such objects can be in good approximation described as the field of inclined rotating magnetic dipole.

In this paper we present calculation of the field of rotating magnetic dipole and we consider also equations of motion of a charged particle in this field. It is shown that each stationary point of the effective potential energy corresponds to particular solution of equations of motion for a charged particle.

2 The field of precessing magnetic dipole

Let us consider the field which is produced by precessing magnetic dipole. We define the law of motion of a dipole moment $\boldsymbol{\mu}$ in the Cartesian coordinate system as follows:

$$\boldsymbol{\mu} = \mu(\sin\alpha\cos\omega t,\ \sin\alpha\sin\omega t,\ \cos\alpha), \tag{1}$$

where $\mu > 0$ and $\omega > 0$ are the magnitude and angular velocity of the dipole respectively, $0 \le \alpha \le \pi$ is the angle between vector $\boldsymbol{\mu}$ and the rotational axis. The general formulae for the field of an arbitrary changing dipole are given, for example, in [8]:

$$\boldsymbol{E} = \frac{(\boldsymbol{n}\times\dot{\boldsymbol{\mu}})}{r^2c} + \frac{(\boldsymbol{n}\times\ddot{\boldsymbol{\mu}})}{rc^2}, \tag{2}$$

$$\boldsymbol{H} = \frac{(\boldsymbol{n}\times(\boldsymbol{n}\times\ddot{\boldsymbol{\mu}}))}{rc^2} + \frac{3\boldsymbol{n}(\boldsymbol{n}\dot{\boldsymbol{\mu}})-\dot{\boldsymbol{\mu}}}{r^2c} + \frac{3\boldsymbol{n}(\boldsymbol{n}\boldsymbol{\mu})-\boldsymbol{\mu}}{r^3}, \tag{3}$$

where c is the speed of light, $\boldsymbol{n} = \boldsymbol{r}/r$ is the unit vector, $\boldsymbol{r}$ is the radius-vector. Field is calculated at time t, and all the quantities on the right side of these equations should be taken at the retarded time $t' = t - \dfrac{r}{c}$.

In a spherical coordinate system (r,θ,φ) the electric field vector has the form:

$$r^3E_r = 0, \tag{4}$$

$$r^3E_\theta = \mu\rho\sin\alpha\left(\rho\sin\tau - \cos\tau\right), \tag{5}$$

$$r^3E_\varphi = -\mu\rho\cos\theta\sin\alpha\left(\rho\cos\tau + \sin\tau\right) \tag{6}$$

where

$$\tau = \omega t' - \varphi, \quad \rho = \frac{r\omega}{c}. \tag{7}$$

Magnetic field vector has components:

$$r^3H_r = 2\mu\left[\sin\alpha\sin\theta(\cos\tau - \rho\sin\tau) + \cos\theta\cos\alpha\right], \tag{8}$$

$$r^3H_\theta = -\mu\left[\cos\theta\sin\alpha(\cos\tau - \rho\sin\tau - \rho^2\cos\tau) - \sin\theta\cos\alpha\right], \tag{9}$$

$$r^3H_\phi = -\mu\sin\alpha(\sin\tau + \rho\cos\tau - \rho^2\sin\tau). \tag{10}$$

One can see that time dependence appears only as composition $\omega t' - \varphi$. It means that the electromagnetic field looks like as the field rotates with angular velocity ω around z-axis. At first glance it would seem that we have a paradox: the linear velocity of motion well away from z-axis is getting more then speed of light. But motion of the field lines does not relate to transfer of matter or field energy. The above equations state only that the electromagnetic field at any point (r, θ, φ) of space is equivalent to the value of field at the point $(r, \theta, \varphi - \delta\varphi)$ at the moment $t - \delta\varphi/\omega$. Evidently, in the far field zone only radiation field remains, which moves radially with the speed of light.

3 Equation of motion of a charged particle

Let us find the Lagrangian for a charged particle in the field under consideration [9]:

$$L = -mc^2\sqrt{1 - \frac{v^2}{c^2}} + \frac{e}{c}\left(\boldsymbol{A}\boldsymbol{v}\right), \tag{11}$$

where $\boldsymbol{v}$ is the particle velocity vector, $\boldsymbol{A}$ is the vector potential. It is easy to prove that the vector potential

$$\boldsymbol{A} = -\frac{(\boldsymbol{n}\times\boldsymbol{\mu})}{r^2} - \frac{(\boldsymbol{n}\times\dot{\boldsymbol{\mu}})}{rc}$$

gives the fields (2) and (3). Substituting spherical components of vectors $\boldsymbol{A}$ and $\boldsymbol{v}$ into Lagrangian (11), we obtain

$$\begin{aligned} L \;=\; & -mc^2\sqrt{1 - \frac{\dot{r}^2 + \dot{\theta}r^2 + \dot{\varphi}^2 r^2\sin\theta^2}{c^2}} \\ & + \frac{e\mu}{cr}\Big\{\dot{\theta}\sin\alpha(\sin\tau + \rho\cos\tau) \\ & + \dot{\varphi}\sin^2\theta\cos\alpha \\ & - \dot{\varphi}\sin\theta\cos\theta\sin\alpha(\cos\tau - \rho\sin\tau)\Big\}. \end{aligned} \tag{12}$$

Further we consider a charged particle to be a nonrelativistic one. Let us introduce a new set of generalized coordinates ρ, θ, ψ with

$$\rho = \frac{r\omega}{c}, \quad \psi = \varphi - \omega t. \tag{13}$$

Actually, this means that we use a co-rotating reference system. As the particle is a non-relativistic one, we restrict our consideration by $\omega r \ll c$ or $\rho \ll 1$. This means that the particle moving around the axis of precession with an angular velocity of about ω is non-relativistic. In this approximation $\tau \approx -\psi$.

Then the nonrelativistic Lagrangian function takes the form

$$\begin{aligned} L \;=\; & \frac{mc^2}{2\omega^2}\left[\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \rho^2(\dot{\psi} + \omega)^2\sin^2\theta\right] \\ & - \frac{e\mu\omega}{c^2\rho}\sin\alpha\Big[(\dot{\psi} + \omega)\sin\theta\cos\theta\cos\psi \\ & + \dot{\theta}\sin\psi\Big] + \frac{e\mu\omega}{c^2\rho}\cos\alpha(\dot{\psi} + \omega)\sin^2\theta. \end{aligned}$$

Substituting this function into Lagrange's equation

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{q}} - \frac{\partial L}{\partial q} = 0, \tag{14}$$

we obtain equations of motion for a charged particle in the field of rotating magnetic dipole

$$\begin{aligned} & \frac{mc^2}{\omega^2}[\ddot{\rho} - \rho\dot{\theta}^2 - \rho(\dot{\psi} + \omega)^2\sin^2\theta] \\ & - \frac{e\mu\omega\sin\alpha}{\rho^2}[\dot{\theta}\sin\psi + \frac{1}{2}(\dot{\psi} + \omega)\sin 2\theta\cos\psi] \\ & \qquad + \frac{e\mu\omega\cos\alpha}{\rho^2}(\dot{\psi} + \omega)\sin^2\theta = 0, \end{aligned} \tag{15}$$

$$\begin{aligned} & \frac{mc^2}{\omega^2}[2\rho\dot{\rho}\dot{\theta} + \rho^2\ddot{\theta} - \frac{1}{2}\rho^2(\dot{\psi} + \omega)^2\sin 2\theta] \\ & + \frac{e\mu\omega\sin\alpha}{\rho^2}[\dot{\rho}\sin\psi - 2\dot{\psi}\rho\sin^2\theta\cos\psi \\ & + \rho\omega\cos 2\theta\cos\psi] \\ & \qquad - \frac{e\mu\omega\cos\alpha}{\rho^2}(\dot{\psi} + \omega)\sin 2\theta = 0, \end{aligned} \tag{16}$$

$$\begin{aligned} & \frac{mc^2}{\omega^2}[\rho^2\ddot{\psi}\sin^2\theta + \rho(\dot{\psi} + \omega)(\rho\dot{\theta}\sin 2\theta + 2\dot{\rho}\sin^2\theta)] \\ & + \frac{e\mu\omega\sin\alpha}{\rho^2}[\sin\theta\cos\theta(\dot{\rho}\cos\psi - \rho\omega\sin\psi) \\ & + 2\rho\dot{\theta}\sin^2\theta\cos\psi] \\ & \qquad + \frac{e\mu\omega\cos\alpha}{\rho^2}(\dot{\theta}\rho\sin 2\theta - \dot{\rho}\sin^2\theta) = 0. \end{aligned} \tag{17}$$

In order to find an integral of motion we multiply Eq. (15) by $\dot{\rho}$, Eq. (16) by $\rho\dot{\theta}$ and Eq. (17) by $\rho\dot{\psi}$ and sum up all the three equations. The left-hand site of resulting equation is a full time derivative. After integration we obtain first order equation

$$\begin{aligned} & \frac{mc^2}{2\omega^2}(\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \rho^2\dot{\psi}^2\sin^2\theta - \omega^2\rho^2\sin^2\theta) \\ & + \frac{e\mu\omega^2\sin 2\theta}{2c^2\rho}\sin\alpha\cos\psi \\ & \qquad - \frac{e\mu\omega^2}{c^2\rho}\cos\alpha\sin^2\theta = \mathcal{E}. \end{aligned} \tag{18}$$

Here $\mathcal{E}$ is a constant. The first term in this expression $K = \frac{mc^2}{2\omega^2}(\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \rho^2\sin^2\theta\dot{\psi}^2)$ is always positive hence, it can be considered as the kinetic energy of the particle. The rest terms are usually referred to as effective potential energy. It can be written as follows:

$$\begin{aligned} V_{ef} = \frac{mc^2}{2}\Big\{ & -\rho^2\sin^2\theta \\ & + \frac{N_\perp\sin 2\theta}{\rho}\cos\psi - \frac{2N_\parallel\sin^2\theta}{\rho}\Big\}, \end{aligned} \tag{19}$$

where

$$N_\perp = \frac{e\mu\omega^2 \sin\alpha}{mc^4}, \quad N_\parallel = \frac{e\mu\omega^2 \cos\alpha}{mc^4}.$$

In this notation Eq. (18) can be represented as:

$$K = \mathcal{E} - V_{ef}. \tag{20}$$

Inequality $\mathcal{E} - V_{ef} \geq 0$ imposes restrictions on possible area of the particle motion.

4 Extremes of the effective potential energy

Let us investigate the effective potential energy. A particle can be in a stable, unstable or indifferent equilibrium at the stationary points of the effective potential energy. The aim of this investigation is to find the stationary points of the function V_{ef}. In order to do so we have to solve the set of equations:

$$\frac{\partial V_{ef}}{\partial q_i} = 0, \tag{21}$$

where $q_i = \rho, \theta, \psi$. This gives a system of three equations

$$-2\rho^3 + 2N_\parallel - 2N_\perp \cot\theta \cos\psi = 0, \tag{22}$$

$$-\rho^3 - 2N_\parallel + 2N_\perp \cot 2\theta \cos\psi = 0, \tag{23}$$

$$N_\perp \sin 2\theta \sin\psi = 0. \tag{24}$$

Equation (24) has next solutions:

$$\psi = 0,\ \pi \text{ and any } \theta; \tag{25}$$

$$\theta = \frac{\pi n}{2},\ (n \in Z) \text{ and any } \psi. \tag{26}$$

4.1 Solution for $\psi = 0, \pi$

Using Eq. (25) we can eliminate the variable ψ from the equations (22) and (23) by substitution $\cos\psi = \varepsilon$, where $\varepsilon = +1$ corresponds to $\psi = 0$ and $\varepsilon = -1$ corresponds to $\psi = \pi$. This results in a system of two equations:

$$-\rho^3 + N_\parallel - \varepsilon N_\perp \cot\theta = 0,$$

$$-\rho^3 \cot\theta - 2N_\parallel \cot\theta + \varepsilon N_\perp (\cot^2\theta - 1) = 0.$$

Excluding $\cot\theta$ from these equations we obtain:

$$2\rho^6 - N_\parallel \rho^3 - N^2 = 0, \tag{27}$$

where $N = \dfrac{e\mu\omega^2}{mc^4}$. This equation has the solution:

$$\rho^3 = \frac{N}{4}\left[\cos\alpha \pm \sqrt{9 - \sin^2\alpha}\right].$$

The sign before the square root is defined by the sign of the charge which is hidden in N, and condition $\rho > 0$. Therefore, the last equation takes the form

$$\rho^3 = \frac{N}{4}\left[\cos\alpha + q\sqrt{9 - \sin^2\alpha}\right], \tag{28}$$

where $q = e/|e| = \pm 1$ is the sign of the charge. And for $\tan\theta$ we find:

$$\tan\theta = -\frac{\varepsilon\left(3\cos\alpha + q\sqrt{9 - \sin^2\alpha}\right)}{2\sin\alpha}. \tag{29}$$

Hence, the solution of equations (22) – (24) for the case $\sin 2\theta \neq 0$ gives two stationary points for a positive charge and two points for a negative charge.

4.2 Solution for $\theta = \dfrac{\pi n}{2}$

It follows from Eqs (22) – (24) that at the axis of rotation ($\theta = 0, \pi$) all the first derivatives from effective potential energy are equal to zero only in the plane $\psi = 0, \pi$ and for any ρ. Which means that on the axis $\theta = 0, \pi$ there are not stationary points.

As to the equatorial plane $\theta = \dfrac{\pi}{2}$, Eqs (22) – (24) have the next solution:

$$\rho = N_\parallel^{\frac{1}{3}}, \quad \cos\psi = 0, \quad N_\parallel > 0. \tag{30}$$

This gives two solutions for $\psi = \pi/2$ and $\psi = 3\pi/2$

According to inequality $N_\parallel > 0$ stated in (30) the sign of the charge and angle α are bound by $e\cos\alpha > 0$, which means that the two above mentioned stationary points correspond to a positive charge if $\alpha < \pi/2$ and to a negative charge if $\alpha > \pi/2$.

Particle located at a stationary point can be in a state of equilibrium. Let us verify whether a particle with initial coordinates $(\rho_i, \theta_i, \psi_i)$ defined by Eqs (28) – (30) and initial zero velocity with respect to the rotating reference frame will be in equilibrium position. Substituting these coordinates and $\dot\rho = \dot\theta = \dot\psi = 0$ in equations of motion (15) – (17) and taking into account that $\varphi = \omega t + \psi$ we obtain identical equalities. Hence, a particle being at rest at one of the stationary points in the co-rotating coordinate system will keep this position. This means that in laboratory reference frame the particle is moving along a circle with constant velocity $v_i = \rho_i c$. Thus, there are at least six particular solutions of equations of motion which describe circuition of the particles in the field of inclined rotating dipole. Positions of the orbits defined by angle ψ and their radius are different for the positive and negative particle. Two of the trajectories are laying in the equatorial plane $z = 0$.

5 Conclusions

We have recorded the components of the magnetic and electric fields of precessing magnetic dipole moment and equations of motion of a charged particle in this field. One first integral of motion was found. This made possible to introduce effective potential energy for the field of precessing magnetic dipole moment. All stationary points of the potential energy were found and it was shown that the stationary points correspond to six particular solutions of equations of motions. These solutions describe circular motion of a particle with a constant velocity.

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ПОЛЕ ПРЕЦЕССИРУЮЩЕГО МАГНИТНОГО ДИПОЛЯ

Исследовано поле прецессирующего магнитного диполя. Получен первый интеграл движения и найдены некоторые частные решения уравнений движения заряженной частицы в этом поле. Рассмотрена эффективная потенциальная энергия заряженной частицы. Показано, что эффективная потенциальная энергия имеет шесть стационарных точек, соответствующих движению заряда по окружности.

Ключевые слова: *магнитный диполь, электромагнитное поле, наклонный ротатор, заряд, уравнения движения.*

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