UDC 537.8

V. Y. Epp, M. A. Masterova

THE POYNTING VECTOR OF AN OBLIQUE MAGNETIC ROTATOR

The density of energy flux of electromagnetic field generated by precessing magnetic dipole at arbitrary distance from the dipole is analyzed. It is shown that in the far field approximation only radial component of the Poynting vector exists, and it represents the intensity of radiation. In the near zone the Poynting vector has not only radial but also sufficient azimuth component. The pattern of field energy flux distribution is constructed. The obtained results can be used at research of dynamics of atmosphere of magnetized celestial bodies.

Keywords: *magnetic dipole, electromagnetic energy flux, electromagnetic radiation, Pointing vector.*

1. Introduction

In this paper we discuss some properties of electromagnetic field, which is generated by varying magnetic dipole.

The field of magnetic dipole moment and motion of charged particles in this field have large practical significance for astrophysics. Magnetic field of the planets and stars can be thought of as dipole in good approximation. Stationary case, when the magnetic moment of a celestial body coincides with the rotation axis, is well studied. Particularly, motion of a charged particle in the Earth magnetic field is studied in details. Solution of equation of motion for charged particle in dipolar magnetic field gives trapping regions for particles, which have energy of limited range. These regions of planets are named radiation belts [1].

There are also well known bodies, which direction of magnetic moment is different from the direction of rotation axis. In this case there is an electric field besides magnetic field around the body. The neutron stars and pulsars are examples of such bodies. Some different models of electric field which is generated in the neighborhood of neutron stars were developed [2]. All these models are based on suggestion that the neutron star is a conducting body. Conducting body, which is rotating in magnetic field, generates a potential difference between the poles and equator. Electromagnetic field of this body differs essentially from the pure dipolar field. At the same time there are celestial bodies, which consist of nonconducting matter and their rotating axis is inclined with respect to the magnetic field axis. Magnetic field of such objects can be in good approximation described as the field of precessing magnetic moment [2].

In this paper we present calculation of the field of precessing magnetic dipole moment and the energy flux in the vicinity of precessing dipole. Usually the energy flux of a source of electro-magnetic field is calculated for the regions far away from a source, in a wave zone. In this zone the energy flux is directed radially outward from source and corresponds to radiation. We have calculated the distribution of energy flux at arbitrary distance from the source. Particularly we have shown that the Poynting vector has not only radial but also an azimuth component in the near zone as well. In other words, energy of the field corotates together with electromagnetic field in the near zone. Radial component of the energy flux takes a large value in wave zone and transforms into radiant flux. Investigation of energy flux in the near zone is of practical importance for celestial bodies which are surrounded by atmosphere of gas.

In this case the electromagnetic field exerts pressure on gas similar to the light pressure. This action can involve the atmosphere matter in corotation with the body.

2. The field of precessing magnetic dipole

Let us consider the field which is produced by precessing magnetic dipole. We define the law of motion of a dipole moment $\vec{\mu}$ in the Cartesian coordinate system as follows:

$$\vec{\mu} = \mu(\sin\alpha\cos\omega t, \sin\alpha\sin\omega t, \cos\alpha), \tag{1}$$

where μ and ω are the module and angular velocity of dipole respectively, α is the angle between vector $\vec{\mu}$ and rotation axis.

The electromagnetic field of magnetic dipole in spherical coordinate system (r, θ, ϕ) can be easily derived from equations [3]:

$$\vec{E} = \frac{[\vec{n}\times\dot{\vec{\mu}}]}{r^2 c} + \frac{[\vec{n}\times\ddot{\vec{\mu}}]}{rc^2}, \tag{2}$$

$$\vec{H} = \frac{[\vec{n}\times[\vec{n}\times\ddot{\vec{\mu}}]]}{rc^2} + \frac{3\vec{n}(\vec{n}\dot{\vec{\mu}}) - \dot{\vec{\mu}}}{r^2 c} + \frac{3\vec{n}(\vec{n}\vec{\mu}) - \vec{\mu}}{r^3}, \tag{3}$$

with $\vec{n}$ – the unit vector in direction of the observer, and $\vec{r} = r\vec{n}$ – the radius vector from dipole to the field point, c is the speed of light. Substituting $\vec{\mu}$ from Eq. (1) we get

$$r^3 E_r = 0, \tag{4}$$

$$r^3 E_\theta = \rho M^* \sin(\omega t' - \psi - \varphi), \tag{5}$$

$$r^3 E_\varphi = -\rho M^* \cos\theta\cos(\omega t' - \psi - \varphi), \tag{6}$$

where

$$t' = t - \frac{r}{c},\ \rho = \frac{r\omega}{c},\ M^* = \mu\sqrt{1+\rho^2}\sin\alpha, \tag{7}$$

$$\sin\psi = \frac{1}{\sqrt{1+\rho^2}}, \ \cos\psi = \frac{\rho}{\sqrt{1+\rho^2}}. \tag{8}$$

It follows from Eqs (4)–(6) that the components of electric field satisfy equation:

$$\left(\frac{E_\theta}{E_{0\theta}}\right)^2 + \left(\frac{E_\varphi}{E_{0\varphi}}\right)^2 = 1, \tag{9}$$

where $E_{0\theta} = \frac{\rho M^*}{r^3}$, $E_{0\varphi} = \frac{\rho M^* \cos\theta}{r^3}$.

Hence, the vector $\vec{E}$ circumscribes an ellipse with semiaxises $E_{0\theta}$ and $E_{0\phi}$ in the plane orthogonal to the radius vector. This ellipse degenerates to circle in direction of precession's axis ($\theta = 0$), but in equatorial plane $\theta = \frac{\pi}{2}$ vector $\vec{E}$ oscillates in meridian plane (parallel to z-axis).

Magnetic field vector has components ($\tau = \omega t' - \varphi$):

$$r^3 H_r = 2\mu[\sin\alpha\sin\theta(\cos\tau - \rho\sin\tau) + \cos\theta\cos\alpha], \tag{10}$$

$$r^3 H_\theta = -\mu[\sin\alpha\cos\theta(\cos\tau - \rho\sin\tau - \rho^2\cos\tau) - -\sin\theta\cos\alpha], \tag{11}$$

$$r^3 H_\varphi = -\mu\sin\alpha(\sin\tau + \rho\cos\tau - \rho^2\sin\tau). \tag{12}$$

It follows from Eqs (10)–(11) that the magnetic field has constant components (in contrast to electric field):

$$H_{rc} = \frac{2\mu}{r^3}\cos\alpha\cos\theta, \tag{13}$$

$$H_{\theta c} = -\frac{\mu}{r^3}\sin\theta\cos\alpha. \tag{14}$$

These components relate to the field of static magnetic dipole if angle of precession is equal to zero. But electric field tends to zero in this case.

One can see that time dependence appears in the formulas for electric and magnetic fields as composition $\omega t' - \varphi$. It means that change of $\omega t'$ is equivalent to changing of φ. In another words geometry of electric and magnetic field looks like as the field rotates with angular velocity ω around z-axis. This conclusion also relates to geometry of the field lines of electric and magnetic fields. At first glance it would seem that we have a paradox: the linear velocity of motion is getting more then light speed well away from z-axis. But motion of the field lines does not relate to transfer of matter or field energy. The above equations state only that the electromagnetic field at any point (r, θ, φ) of space is equivalent to the value of field at the point of space (r, θ, φ–δφ) in the moment t–δφ/ω. Evidently that in the wave zone only radiation field exists, which moves radially with the light speed.

3. Energy flux density of field near the precessing dipole

Knowing the field of the dipole one can calculate the value and direction of energy transfer by electromagnetic field using the Poynting vector. In the limit $r \to \infty$ we expect the well known case of electromagnetic radiation – Poynting vector, decreasing as $1/r^2$, and directed along the radius vector $\vec{r}$.

In the near zone the energy transfer is more complicated and has not been studied until now. If the celestial body is surrounded by atmosphere of absorbing matter, the field energy flow can affect the flow of matter. Namely, the field can transfer part of its momentum to matter and exert a pressure on it. Thus, dynamics of atmosphere near rotating magnetic body can be influenced by the energy flow of electromagnetic field in the near zone. In this section we construct a picture of Poynting vector field lines at arbitrary distance from the rotating dipole.

Vector of energy flux density, named as Poynting vector is defined by formula [4]:

$$\vec{P} = \frac{c}{4\pi}\left[\vec{E}\times\vec{H}\right]. \tag{15}$$

Let us find components of Poynting vector in spherical coordinate system. Using Eqs (4)–(6) for electric field and Eqs (10)–(11) for magnetic field we obtain:

$$\begin{aligned} P_r = P_0\rho\{&\rho^3\sin^2\alpha(\sin^2\tau + \cos^2\theta\cos^2\tau) - \\ &-\rho^2\sin^2\alpha\sin 2\tau\sin^2\theta + \rho\sin^2\alpha\cos 2\tau\sin^2\theta + \\ &+\frac{1}{2}\sin^2\alpha\sin 2\tau\sin^2\theta + \\ &+\frac{1}{4}\sin 2\alpha\sin 2\theta(\rho\cos\tau + \sin\tau)\}, \end{aligned} \tag{16}$$

$$\begin{aligned} P_\theta = &P_0\rho\sin\alpha\cos\theta\times \\ &\times[\sin\alpha\sin\theta(\rho^2\sin 2\tau - \rho\cos 2\tau - \sin 2\tau) - \\ &-2\cos\alpha\cos\theta(\rho\cos\tau + \sin\tau)], \end{aligned} \tag{17}$$

$$\begin{aligned} P_\varphi = &2P_0\rho\sin\alpha[\sin\alpha\sin\theta(\rho^2\sin^2\tau - \rho\sin 2\tau + \\ &+\cos^2\tau) - \cos\alpha\cos\theta(\rho\sin\tau - \cos\tau)], \end{aligned} \tag{18}$$

where $P_0 = \frac{c\mu^2}{4\pi r^6}$.

Energy flux is fast oscillating function at any point of space. Of practical interest is the value of energy flux averaged over period. Eqs (16)–(18) averaged over time give a stationary pattern of field energy flux distribution:

$$P_r = \frac{\mu^2\omega^6}{8\pi c^5\rho^2}\sin^2\alpha(1 + \cos^2\theta),$$

$$P_\theta = 0, \tag{19}$$

$$P_\varphi = \frac{\mu^2\omega^6}{4\pi c^5\rho^5}\sin^2\alpha\sin\theta(\rho^2+1).$$

Let us check that in the limit $r \to \infty$ Eq. (19) gives the known expression for radiation intensity of dipole.

Radiation intensity I per unit of solid angle Ω is given by expression:

$$\frac{dI}{d\Omega} = \frac{1}{4\pi c^3}\left[\ddot{\vec{\mu}} \times \vec{n}\right]^2. \qquad (20)$$

Substituting μ from Eq. (1) and averaging over time we obtain:

$$\frac{dI}{d\Omega} = \frac{\mu^2\omega^4\sin^2\alpha(1+\cos^2\theta)}{8\pi c^3}. \qquad (21)$$

On the other hand, we find the radiation intensity as a limit of Poynting vector when $r \to \infty$. We can see from Eq. (19) that only radial component of vector $P_\infty = P\big|_{r\to\infty} = P_r$ is left in this limit. Substituting $\rho = \frac{r\omega}{c}$, we obtain:

$$P_\infty = \frac{\omega^4\mu^2}{8\pi r^2 c^3}\sin^2\alpha(1+\cos^2\theta). \qquad (22)$$

If we insert this term into formula for radiation intensity $dI = Pr^2 d\Omega$ we obtain Eq. (21) again. Thus, Eqs (19) define the vector $\vec{P}$ for any distance from the dipole and take a correct form in the limit $r \to \infty$.

It follows from Eqs (19) that the azimuth component of Poynting vector dominates in the region $\rho \ll 1$ and radial component dominates in case $\rho \gg 1$. Let us construct the field lines of vector $\vec{P}$. It follows from $P_\theta = 0$ that the line which is running through arbitrary point of space is located on a cone surface. The vertex of this cone coincides with the origin of coordinate system and its axis coincides with z-direction. Let us construct a projection of the energy flux line on xy-plane (Fig. 1).

We write equation for the flux line in polar coordinates (R, φ), with φ measured from the x-axis. The coordinate R is projection of radius vector on xy-plane: $R = r\sin\theta$.

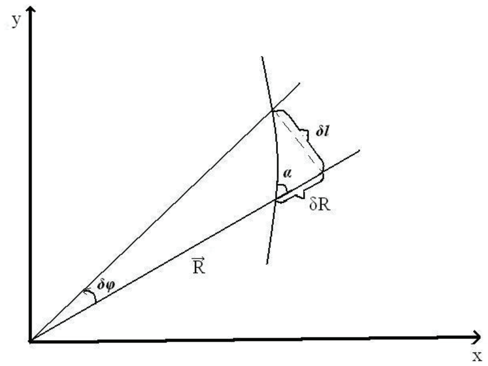

Fig. 1. Slope of the energy flux line to vector $\vec{R}$

We see from Fig. 1 that the angle α is defined by relation:

$$\mathrm{tg}\alpha = \frac{\delta l}{\delta R} = \frac{R\delta\varphi}{\delta R}. \qquad (23)$$

On the other hand:

$$\mathrm{tg}\alpha = \frac{P_{\varphi xy}}{P_{r\,xy}}, \qquad (24)$$

where $P_{\varphi xy}$ and P_{rxy} are projections of components P_φ and P_r on xy-plane:

$$P_{\varphi xy} = P_\varphi,\; P_{r\,xy} = \sin\theta\, P_r. \qquad (25)$$

Eqs (23) and (24) with account of Eqs (19) result in differential equation for the field line of vector $\vec{P}$:

$$\frac{d\varphi}{d\rho} = 2\frac{1+\rho^2}{\rho^4(1+\cos^2\theta)}. \qquad (26)$$

Solution of this equation gives the Poynting vector field lines on the xy-plane:

$$\varphi = -\frac{2}{(1+\cos^2\theta)}\left(\frac{1}{\rho} + \frac{1}{3\rho^3}\right) + \varphi_0, \qquad (27)$$

where φ_0 is the constant of integration. Different field lines correspond to different value of φ_0. As is seen from Eq. (27) one line differs from another by turn around z-axis.

Figs 2–4 show projection of a single flux line on the xy-plane in coordinates (φ, ρ).

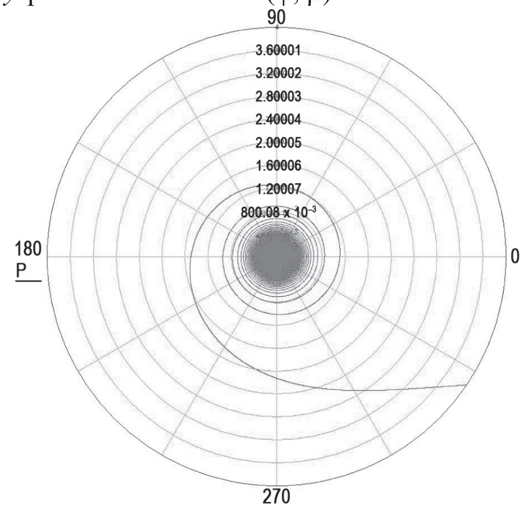

Fig. 2. Power lines pattern for $\theta = \frac{\pi}{2}$, r changes from 0 to 4

Thus, the vector $\vec{P}$ field line is a helix which is wrapped around a cone with its vertex at the dipole and its axes coincides with the z-axis. Pitch of the helix is small in region of small ρ and the flux line is close to a circle in this case. As the coordinate ρ increases, so does the pitch of helix, and at high ρ the flux line approximates to the radial direction. If we compare graphics for different θ, we must keep in mind that they are constructed in coordinates (φ, ρ), but the real radial coordinate is R, which is related to ρ

by equation

$$\rho = \frac{\omega R}{c \sin\theta}. \tag{28}$$

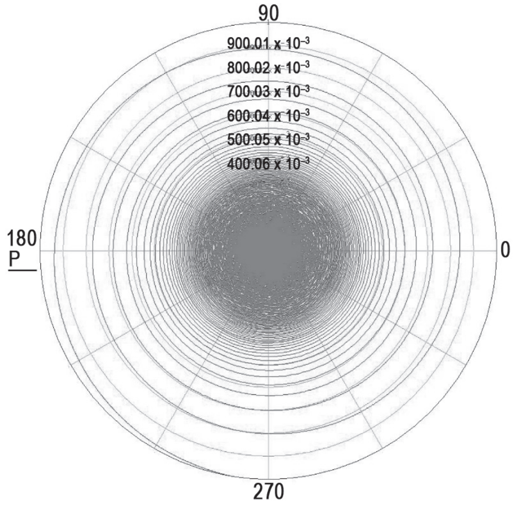

Fig. 3. Power lines pattern for $\theta = \frac{\pi}{2}$, *r* changes from 0 to 1

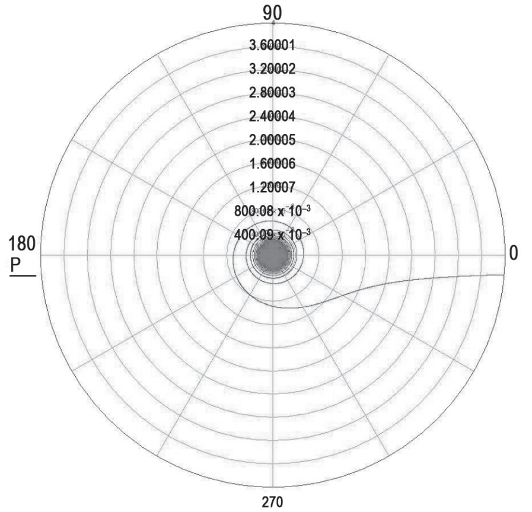

Fig. 4. Power lines pattern for $\theta = \frac{\pi}{6}$, *r* changes from 0 to 4

In other words graphics are constructed with a scale-factor $\frac{\omega}{c}\sin\theta$.

The value of ρ is non-dimensional distance from the coordinate origin along the generatrix of the cone which contains the flux line.

We can estimate dependence of the pitch of the helix on distance ρ in the following way. In the region $\rho \ll 1$ the pitch of helix λ is much less than ρ. Hence, in Eq. (26), which we write down as

$$d\varphi = 2\left(\frac{c}{\omega R}\right)^3 \frac{\sin\theta\left(\sin^2\theta + \frac{\omega^2 R^2}{c^2}\right)}{1+\cos^2\theta}\frac{dR}{R}. \tag{29}$$

one can put $d\varphi \approx 2\pi,\ dR \approx \lambda$. Then

$$\lambda = \pi R\left(\frac{\omega R}{c}\right)^3 \frac{1+\cos^2\theta}{\sin\theta\left(\sin^2\theta + \frac{\omega^2 R^2}{c^2}\right)}. \tag{30}$$

We see that at small ρ the pitch of helix rises as R^4. It also rises when angle θ decreases. And it tends to infinity if $\theta \to 0$ – the helix degenerates in a straight line which coincides with *z*-axis. The same follows from Eqs (19), which show that only radial component of $\vec{P}$ is present at θ = 0.

4. Summary

We have analyzed average density of electromagnetic energy flux generated by the field of precessing magnetic dipole. Expression for Poynting vector is derived for arbitrary distance from the source of the field. It is shown that in the limit of far zone the obtained formulaes give the expected result – the energy flux of the field is the electromagnetic radiation, which drops off with distance *r* as $1/r^2$, and the radial component of Poynting vector prevails.

In the near zone, at distances comparable to the wave length, the azimuth component of Poynting vector is much greater then the radial one. It means the energy flow at any point in the near zone is almost orthogonal to the radius vector of the point and orthogonal to the axis of rotation. This energy flow can involve in corotation the atmosphere of magnetized rotating celestial body. The results can be used for analysis of magnetosphere and dynamics of atmosphere of such celestial objects. The intrinsic magnetic field of a planet can also affect the global electromagnetic processes in the atmosphere of the planet such as thunderstorm activity [5].

Acknowledgment

This work has been supported by the grant for LRSS, project No 3558.2010.2.

References

1. Holmes-Siedle A. G., Adams L. Handbook of Radiation Effects. Oxford University Press, England, 2002.
2. Michel F. C. Theory of Neutron Star Magnetospheres. L.: Chicago Press, 1991. P. 287.
3. Feynman R. P., Leighton R. B., Sands M. The Feynman Lectures on Physics. Vol. 2 . Addison-Wesley, 1964.
4. Landau L. D., Lifshitz E. M. The Classical Theory of Fields. 4th ed., Pergamon, New York, 1975.
5. Gorbatenko V. P. About spatial variations of thunder days and density of lightning discharges to the ground // Tomsk State Pedagogical University Bulletin. 2000. Issue 2. P. 39–42. (in Russian).

Epp V. Y.
Tomsk State Pedagogical University.
Ul. Kievskaya, 60, Tomsk, Russia, 634061.

Masterova M. A.
Tomsk State Pedagogical University.
Ul. Kievskaya, 60, Tomsk, Russia, 634061.

Received 14.03.2011.

В. Я. Эпп, М. А. Мастерова

ВЕКТОР УМОВА-ПОЙНТИНГА НАКЛОННОГО МАГНИТНОГО РОТАТОРА

Проводится исследование потока энергии электромагнитного поля в окрестности прецессирующего магнитного диполя. Вычислено распределение потока энергии поля на любых расстояниях от источника поля. Показано, что на больших расстояниях полученные формулы переходят в известные выражения для интенсивности излучения диполя. В ближней зоне вектор Умова-Пойнтинга имеет не только радиальную но и существенную азимутальную составляющую. Построены силовые линии вектора Умова-Пойнтинга для распределения потока энергии электромагнитного поля. Полученные результаты могут быть использованы при исследованиях динамики атмосферы небесных тел, обладающих достаточно сильным магнитным полем.

Ключевые слова: *магнитный диполь, электромагнитное поле, поток энергии, электромагнитное излучение, вектор Пойнтинга.*

Эпп В. Я., доктор физико-математических наук, профессор.
Томский государственный педагогический университет.
Ул. Киевская, 60, Томск, Россия, 634061.

Мастерова М. А., аспирант.
Томский государственный педагогический университет.
Ул. Киевская, 60, Томск, Россия, 634061.