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A SPECTRUM OF THE DIRAC OPERATOR WITH AN EXTERNAL YANG-MILLS GAUGE FIELD ON DE SITTER SPACE

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The Dirac operator with an external Yang–Mills gauge field is considered on de Sitter space in terms of a noncommutative integration method related to the orbit method in the Lie group theory. A Yang–Mills field is presented for which the de Sitter group serves as the symmetry group of the Dirac operator. A spectrum of the Dirac operator with the Yang–Mills field is calculated in explicit form.

Keywords: *the Dirac equation, noncommutative integration, de Sitter space.*

1 Introduction

Constructing solutions to the relativistic wave equations on curved spaces with external fields is an inevitable stage in studying effects in quantum field theory and cosmology [1, 2].

The integration of relativistic wave equations with external fields is usually based on the classical method of separation of variables (SoV) [3–5]. The traditional SoV method involves a complete set of commuting observables that is a set of commuting symmetry operators whose eigenvalues completely specify the state of a quantum system [6].

A new method of exact integration of linear partial differential equations based on noncommutative sets of symmetry operators other than the SoV method was proposed in Ref. [7]. This method provides new possibilities to study the relativistic quantum wave equations and constructing solutions different from ones obtained by the SoV method.

In this work, we consider a Yang-Mills potential for which the generators of the de Sitter group form a noncommutative symmetry algebra for the Dirac operator. In the framework of the noncommutative integration method, we calculate a spectrum for the Dirac operator with the corresponding Yang-Mills potential.

2 Symmetry of the Dirac operator in de-Sitter space

Consider the de Sitter space M with the de Sitter isometry group of transformations $G = SO(1,4)$ and an isotropy Lorentz subgroup $H = SO(1,3)$. The space M is topologically isomorphic to $R^1 \times S^3$ and has a constant positive curvature.

The de Sitter group $SO(1,4)$ is a rotation group of a 5-dimensional pseudo-Euclidean space with the metric $G_{AB} = \mathrm{diag}(1,-1,-1,-1,-1)$. The algebra $\mathfrak{g} = \mathfrak{so}(1,4)$ of the de Sitter group can be defined in terms of the basis $\{E_{AB} \mid A < B\}$ by the following commutation relations:

$$[E_{AB}, E_{CD}] = G_{AD}E_{BC} - G_{AC}E_{BD} + \\ + G_{BC}E_{AD} - G_{BD}E_{AC},$$

where $A, B, C, D = 1, \dots, 5$. The basis E_{AB} can be written as $(a, b = 1, \dots, 4)$

$$E_{ab} = e_{ab},\ (a < b), \quad E_{a5} = e_a/\varepsilon, \quad [e_a, e_b] = \varepsilon^2 e_{ab}.$$

Here the basis e_{ab} forms an isotropy subalgebra $\mathfrak{h} = \mathfrak{so}(1,3)$, and ε is a parameter defining the curvature of de Sitter space, $R = 12\varepsilon^2$. Define canonical coordinates of the second type for a Lie group G as

$$g(t, x, y, z, h_1, \dots, h_6) = e^{h_6 e_{34}} e^{h_5 e_{24}} e^{h_4 e_{23}} e^{h_3 e_{14}} e^{h_2 e_{13}} \times \\ \times e^{h_1 e_{12}} e^{z e_4} e^{y e_3} e^{x e_2} e^{t e_1}.$$

Here $\mathbf{x} = (t, x, y, z)$ are local coordinates on the de Sitter space M and h_{ab} are local coordinates on the isotropy subgroup H. In terms of these coordinates the line element of M takes the form

$$ds^2 = g_{ab}(\mathbf{x}) d\mathbf{x}^a d\mathbf{x}^b = \rho^2(x, y, z) dt^2 - \\ - \cos^2 \varepsilon y \cos^2 \varepsilon z dx^2 - \cos^2 \varepsilon z dy^2 - dz^2,$$

where $\rho(x, y, z) = \cos \varepsilon x \cos \varepsilon y \cos \varepsilon z$.

Let V_K be a set of vector fields on M transforming according to the fundamental representation of an N-dimensional gauge Lie group K. Denote by V_Ψ a space of spinor fields on M. A multiplet of N–spinor fields on M can be considered as a space $C^\infty(M, V)$ of functions on M taking values in a linear space $V = V_\Psi \otimes V_K$.

Let us write down the Dirac operator in the space $C^\infty(M, V)$ as follows [8]:

$$\mathcal{D} = i\gamma^a(\mathbf{x})[\nabla_a + \Gamma_a(\mathbf{x}) + i\kappa A_a(\mathbf{x})]. \tag{1}$$

Here ∇_a is the covariant derivative corresponding to the Levi-Civita connection on M, κ is a coupling constant. The Dirac gamma matrices, $\gamma^a(x)$, satisfy the condition

$$\{\gamma_a(\mathbf{x}), \gamma_b(\mathbf{x})\} = 2g_{ab}(\mathbf{x})E_4, \tag{2}$$

where E_4 denotes an identity matrix. The spinor connection $\Gamma_a(\mathbf{x})$ satisfies the conditions $[\nabla_a + \Gamma_a(\mathbf{x}), \gamma_b(\mathbf{x})] = 0$, $\operatorname{Tr}\Gamma_a(\mathbf{x}) = 0$ and can be presented in explicit form as [9]:

$$\Gamma_a(\mathbf{x}) = -1/4(\nabla_a\gamma_b(\mathbf{x}))\gamma^b(\mathbf{x}).$$

Let us say that the Dirac operator $\mathcal{D}$ on the de Sitter space M admits an external gauge field if the motion group of M is the symmetry group of the Dirac operator.

The external gauge field potentials for which the de Sitter group is the symmetry group of the Dirac operator (1) are determined by the equation

$$[\mathcal{D}, \hat{X}] = 0, \tag{3}$$

where the operators $\hat{X}$ are *generators* of a transformation group G acting on $C^\infty(M, V)$. Such a potential is given as

$$\begin{aligned} A_1 &= -i\varepsilon\sin\varepsilon x\Lambda_{12} + \varepsilon\cos\varepsilon x\sin\varepsilon y\Lambda_{13} + \\ &+\varepsilon\cos\varepsilon x\cos\varepsilon y\sin\varepsilon z\Lambda_{14}, \\ A_2 &= \varepsilon\sin\varepsilon y\Lambda_{23} + \varepsilon\cos\varepsilon y\sin\varepsilon z\Lambda_{24}, \\ A_3 &= \varepsilon\sin\varepsilon z\Lambda_{34}, \quad A_4 = 0, \end{aligned} \tag{4}$$

where $\Lambda_{ab} = -\frac{i}{4}[\hat{\gamma}^a, \hat{\gamma}^b]$ are representation generators of the isotropy subgroup H in a gauge space $\mathbb{R}^4$, $\hat{\gamma}^a$ are the standard Dirac gamma matrices. The isotropy subgroup H is the gauge group. Then the Dirac operator (1) with the potential (4) reads

$$\begin{aligned} \mathcal{D} = &-i\bigg(\frac{\hat{\gamma}^1}{\rho}\partial_t + \hat{\gamma}^2\frac{\cos\varepsilon x}{\rho}\partial_x + \hat{\gamma}^3\frac{\cos\varepsilon x\cos\varepsilon y}{\rho}\partial_y - \\ &-\hat{\gamma}^4\partial_z - \varepsilon\hat{\gamma}^2\frac{\sin\varepsilon x}{2\rho} - \varepsilon\hat{\gamma}^3\frac{\tan\varepsilon y}{\cos\varepsilon z} - \frac{3\varepsilon}{2}\hat{\gamma}^4\tan\varepsilon z\bigg)\otimes\mathbb{E}_4 + \\ &+\varepsilon\kappa\bigg(\hat{\gamma}^1\otimes\bigg[\tan\varepsilon z\Lambda_{14} + \frac{\sin\varepsilon x}{\rho}\Lambda_{12} + \frac{\tan\varepsilon y}{\cos\varepsilon z}\Lambda_{13}\bigg] + \\ &+\hat{\gamma}^2\otimes\bigg[\tan\varepsilon z\Lambda_{24} + \frac{\tan\varepsilon y}{\cos\varepsilon z}\Lambda_{23}\bigg] + \hat{\gamma}^3\otimes\Lambda_{34}\tan\varepsilon z\bigg). \end{aligned}$$

It can be verified that the motion group of M is the symmetry group of the Dirac operator only if $\kappa = 1$.

3 Noncommutative integration method

Consider the equation

$$\mathcal{D}\psi = \lambda\psi, \quad \psi \in C^\infty(M, V), \tag{5}$$

with the potential (4) and apply the noncommutative integration method [7]. Note that in the framework of this method, the condition of integer-valued orbits [10] allows us to consider (5) as an eigenvalue problem. Equation (5) results in the following system of equations on the Lie group G:

$$i\hat{\gamma}^a\eta_a(g)\psi(g) = \lambda\psi(g), \tag{6}$$

$$(i\eta_{ab}(g) - \Lambda_{ab}\otimes\mathbb{E}_4 + \mathbb{E}_4\otimes\Lambda_{ab})\psi(g) = 0. \tag{7}$$

By means of the noncommutative reduction [7, 11] of the system (6),(7), we obtain

$$i\hat{\gamma}^a\ l_a(q, j)\hat{\psi}(q) = \lambda\hat{\psi}(q), \tag{8}$$

$$(il_{ab}(q, j) - \Lambda_{ab}\otimes\mathbb{E}_4 + \mathbb{E}_4\otimes\Lambda_{ab})\hat{\psi}(q) = 0. \tag{9}$$

Here the function $\hat{\psi}(q)$ is a Fourier transform of $\psi(g)$, and $l_A(q)$ are operators of the so called λ–representation of the de Sitter algebra and have the form

$$l_A(q, j) = \alpha^i_A(q)\partial_{q^i} + i\chi_A(q, j), \quad A = (a, ab). \tag{10}$$

From equation (9) we find the derivatives of $\hat{\psi}(q)$, substitute them into (8), and come to the algebraic system of equations

$$-j\hat{\gamma}^0\otimes\mathbb{E}_4 - \varepsilon\sum_{a=2}^{4}\hat{\gamma}^a\otimes\Lambda_{1a} = \lambda\mathbb{E}_4\otimes\mathbb{E}_4. \tag{11}$$

From (11) we finally have

$$\begin{aligned} &\left[\lambda^2 - (2j - 3i\varepsilon)^2 - 9\varepsilon^2\kappa^2\right]^6 \times \\ &\times\left[\lambda^2 - (2j - 3i\varepsilon)^2 + \varepsilon^2\kappa^2\right]^2 = 0. \end{aligned} \tag{12}$$

Note that λ can take real values only if $j = 0$. From (12) we see that if the gauge field disappears, $\kappa = 0$, then the spectrum has an eightfold degeneracy. Switching on the field ($\kappa = 1$) lifts the degeneracy in part. Then one can see that the field (4) preserves the Killing symmetry of the Dirac operator in the sense of relation (3), but violates the spin symmetry.

4 Conclusion

We consider a Yang-Mills potential (4) for which the generators of the de Sitter group are a non-commutative symmetry algebra for the Dirac operator (1). Following the noncommutative integration method, we examine an eigenvalue problem of the Dirac operator. The eigenvalues are determined by the algebraic condition (11). The parameter j enumerates singular orbits of the coadjoint representation of the de Sitter algebra.

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СПЕКТР ОПЕРАТОРА ДИРАКА С ВНЕШНИМ КАЛИБРОВОЧНЫМ ПОЛЕМ ЯНГА–МИЛЛСА В ПРОСТРАНСТВЕ ДЕ СИТТЕРА

Оператор Дирака с внешним калибровочным полем Янга-Миллса рассматривается на пространстве де Ситтера в рамках метода некоммутативного интегрирования, связанного с методом орбит в теории групп Ли. В явном виде вычислен спектр оператора Дирака с полем Янга–Миллса, для которого группа де Ситтера является группой симметрии уравнения Дирака.

Ключевые слова: *уравнение Дирака, некоммутативное интегрирование, пространство де Ситтера.*

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