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THE USE OF MODIFIED HILL'S VARIABLES IN THE METHOD OF AVERAGING

In this paper the modified Hill's canonical variables are introduced. They are v, G, H; r, g, h, where r is the distance to the primary for Keplerian orbit ; $v = dr / dt$; $G = \sqrt{a\mu(1-e^2)}$ and $H = G \cos i$ are Delaunay's variables; $g = \omega$ is the argument of the periapsis; $h = \Omega$ is the longitude of the ascending node. We show that the modified Hill's canonical variables are useful in some analytical theories of motion where it is need to avoid expansion of the perturbations in powers of eccentricity and where it is advantageous to use the eccentric anomaly as independent variable. As an example, we consider the single-averaged analytical model in the frame of the restricted three-body problem for satellite case.

Key words: *celestial mechanics, perturbation theory, averaging method, Lie transformations, Hill's canonical variables, restricted three-body problem, satellites dynamics.*

1. Introduction

The main goal of this study is to obtain the algorithm for the most compact presentation of analytical theory of motion in closed form (without expansion of the perturbation in powers of the eccentricity). K. Aksnes [1] has shown that the resulting analytical theory may be more compact, if all formulas of problem are expressed in terms of r and v (radial velocity $v = \dot{r} = dr/dt$) instead of f (true anomaly) or u (eccentric anomaly). Our work is based on this idea.

A motivation for this research is the study of the secular dynamics of irregular satellites of the outer planets. It is known, that the development of analytical perturbation methods for these objects is very difficult problem. The orbital characteristics and the complex interaction between the different degrees of freedom of such dynamical systems require to develop high-order analytical models. Many analytical and semianalytical theories for irregular satellites motion are constructed [2–6]. But as shown in [4], the long-term dynamics of such satellites is only partially understood. In the present paper we develop a new approach to construction of high-order analytical model for motion of a satellite in the restricted three-body problem, which is based on the use of the modified Hill's canonical variables v, G, H; r, g, h. This variables are important in our problem at the stage of elimination of fast variable u. Therefore we consider the single – averaged analytical model in this paper.

2. Canonical variables

In our work we avoid expansion of the perturbations in powers of the eccentricity and inclination. To carry such theories in analytical form to high order, it is necessary to perform the calculation on computer. The algorithm of automated, closed form integration of formulas in elliptic motion was developed by William H. Jefferys [7] for Hill's canonical variables:

$$\dot{r},\quad G,\quad H,\quad r,\quad \theta,\quad h, \text{(HCV)} \tag{1}$$

where $\theta = f + \omega$ is argument of latitude; f is true anomaly; HCV is Hill canonical variables.

Hill's variables are convenient for developments of the oblateness-type perturbations. But if perturbations by third body are to be treated it is advantageous to use the eccentric anomaly instead of the true anomaly.

In the present work we introduce another set of canonical variables:

$$\dot{r},\quad G,\quad H,\quad r,\quad g,\quad h \text{ (MCV)} \tag{2}$$

where, MCV are modified canonical variables.

With help of Pfaff's theory [8] in the expanded phase space it can be shown that variable (2) are canonical. Pfaff's form of a canonical system

$$\Phi = p\,dq - H\,dt \tag{3}$$

is a simple criterion to test canonicity. From fundamental Pfaffian [8] of perturbed two-body problem it is easy to receive for our case the following form:

$$\Phi = \dot{r}\,dr + \sqrt{\mu a(1-e^2)}\,d\omega + \\ + \sqrt{\mu a(1-e^2)}\cos i\,d\Omega - E_0 dt. \tag{4}$$

The symbol E_0 in (4) represents the total energy of the system.

For Delaunay elements:

$$L, G, H, l, g, h \text{ (DCV)} \tag{5}$$

the Pfaffian is written in a form

$$\Phi = \sqrt{\mu a}\,dl + \sqrt{\mu a(1-e^2)}\,d\omega + \\ + \sqrt{\mu a(1-e^2)}\cos i\,d\Omega - E_0 dt = \\ = L\,dl + G\,dg + H\,dh - E_0 dt. \tag{6}$$

Several remarks may now make about the properties of canonical variables: DCV, HCV and MCV.

DCV(Delaunay's variables) are singular for zero eccentricity orbits, where the argument of the periapsis is not defined. DCV are the action-angle variables. Usually the Hamiltonian is treated implicitly as function of canonical Delaunay elements, but it is not convenient if it is necessary to perform the calculations on computers.

HCV (Hill's variables) are nonsingular. HCV are not the action-angle variables. Canonical variable $\theta = f + \omega$ contains true anomaly. But if perturbations by third body are to be treated it is needed transformation to the eccentric anomaly.

MCV (Modified variables) are singular for zero eccentricity orbits and are not the action-angle variables. The Hamiltonian is completely expressed in terms of MCV.

In present work we use Lie transformations [9–11] for elimination of the short period terms from disturbing function. In this method a symplectic change of variables is given not by a classical generating function, but by a generator $W(\dot{r},G,H,r,g,h,\varepsilon)$. It is a function such that the shift of the canonical system with Hamiltonian function W by time ε along the trajectories produces the required transformation. Generator $W(\dot{r},G,H,r,g,h,\varepsilon)$ depends only on the new variables and algorithm of Lie transformation is reduced to calculations of Poisson brackets. Therefore this method is convenient for high order calculations and as shown in [11], it permits to use the variables which are not the action-angle ones. In conclusion we mention that, due to the invariant character of Poisson brackets, we may use occasionally Delaunay's variables in the computations.

3. Single-averaged analytical model

We consider the restricted three-body problem comprising a massless satellite orbiting a planet of mass m_0 and perturbed by the Sun, with mass m'. The geometry of this problem is shown in Fig. 1.

It is assumed that the planet is fixed in the center of the reference system *X*, *Y*, *Z*. The satellite is in the orbit which orbital elements are *a* (semimajor axis), *e* (eccentricity), *i* (inclination), ω (argument of periapsis), Ω (longitude of the ascending node) and *n* (mean motion of a satellite) given by the expression $n^2a^3 = Gm_0$, *G* is gravitational constant. The perturbing body, with mass m', is in a elliptical orbit with radius-vector $\mathbf{r}'$. *S* is the angle between radius-vectors $\mathbf{r}$ and $\mathbf{r}'$.

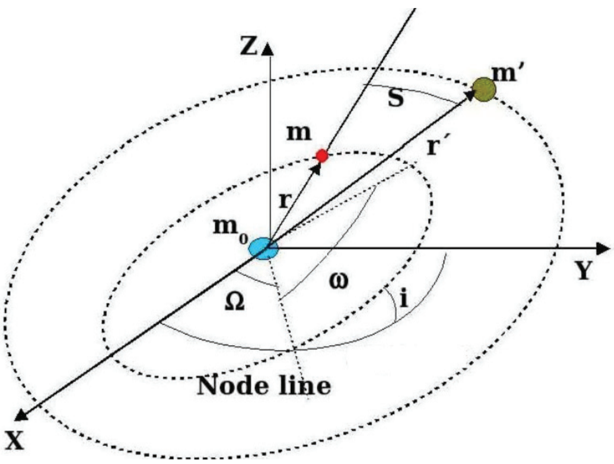

Fig. 1. The geometry of the restricted three-body problem

Using the traditional expansion in Legendre polynomials P_i the disturbing function is given by

$$R = \frac{G(m_0+m')}{r'} \sum_{i=2}^{\infty} \left(\frac{r}{r'}\right)^i P_i\left[\cos(S)\right], \tag{7}$$

where $r' \gg r$ and

$$\cos S = \cos(f+\omega)\cos(\Omega-l') - \sin(f+\omega)\sin(\Omega-l')\cos i. \tag{8}$$

Here l' is longitude of the perturbing body.

The Hamiltonian function in modified variables is given by

$$F = F_{00} + \varepsilon F_{01} + \varepsilon^2 F_{02}, \tag{9}$$

$$F_{00} = \frac{k^2}{2\alpha^2}, \quad F_{01} = H, \tag{10}$$

$$F_{02} = \sum_{n=2}^{\infty} (r)^n P_n\left[\cos(S)\right]. \tag{11}$$

$$\cos S = \cos(f+g)\cos h - \sin(f+g)\sin h\cos i, \tag{12}$$

where $g=\omega$, $h = \Omega - l'$.

The canonical system for variables $\dot{r}$, G, H, r, g, h is represented in form of the Pfaff's equation [8] for variables *v*, *s*, *c*, *r*, *g*, *h*.

$$v=\dot{r}, \ G=\alpha k s, \ \alpha=\sqrt{a}, \ k=\sqrt{\mu},$$

$$\mu = Gm_0, \ s=\sqrt{1-e^2}, \ H = \alpha k s\cos i = \alpha k c. \tag{13}$$

The equations of motion have the form

$$\frac{d\dot{r}}{dt} = -\frac{\partial F}{\partial r}, \quad \frac{ds}{dt} = -\frac{1}{k\alpha}\frac{\partial F}{\partial g}, \quad \frac{dc}{dt} = -\frac{1}{k\alpha}\frac{\partial F}{\partial h};$$

$$\frac{dr}{dt} = \frac{\partial F}{\partial \dot{r}}, \quad \frac{dg}{dt} = \frac{1}{k\alpha}\frac{\partial F}{\partial s}, \quad \frac{dh}{dt} = \frac{1}{k\alpha}\frac{\partial F}{\partial c}. \tag{14}$$

The Poisson brackets are

$$\{W, f_{ij}\} = \frac{\partial W}{\partial r}\frac{\partial f_{ij}}{\partial v} - \frac{\partial W}{\partial v}\frac{\partial f_{ij}}{\partial r} + \frac{1}{k\alpha}\left(\frac{\partial W}{\partial g}\frac{\partial f_{ij}}{\partial s} - \frac{\partial W}{\partial s}\frac{\partial f_{ij}}{\partial g}\right) + \frac{1}{k\alpha}\left(\frac{\partial W}{\partial h}\frac{\partial f_{ij}}{\partial c} - \frac{\partial W}{\partial c}\frac{\partial f_{ij}}{\partial h}\right), \tag{15}$$

where f_{ij} are arbitrary function in terms (13), *W* is the generator of the Lie transformations.

We assume that the functions to be calculated are sums of the form

$$f_{ij}^{(1)} = \sum_{\sigma,\chi,\xi} \phi_\sigma(v,r,s,c,\alpha,k) \frac{SIN}{COS}(\chi g + \xi h) \tag{16}$$

or

$$f_{ij}^{(2)} = \sum_{\sigma,\tau,\chi,\xi} \varphi_\sigma(s,c,\alpha,k) \frac{SIN}{COS}(\tau u + \chi g + \xi h). \tag{17}$$

In (16) – (17) u is eccentric anomaly, k and α are parameters, $k = n\alpha^3$, where n is mean motion of a satellite.

The functions $f_{ij}^{(1)}$ and $f_{ij}^{(2)}$ are connected by the relations

$$v = \frac{\alpha k\sqrt{1-s^2}\sin(u)}{r}, \quad r = \alpha^2(1-\sqrt{1-s^2}\cos u),$$

$$\sin u = \frac{r\,v}{\alpha k\sqrt{1-s^2}}, \quad \cos u = \frac{\left(1-\frac{r}{\alpha^2}\right)}{\sqrt{1-s^2}}. \tag{18}$$

The functions (16) can be regarded as the polynomials in v and r. They are the most compact expressions. In our algorithm functions $f_{ij}^{(1)}$ in complex exponential form are used for calculation of the Poisson brackets on computer.

Consider the formal procedure of averaging the polynomials $f_{ij}^{(1)}$. Here we discuss a question relating to the averaging function $f_{ij}^{(1)}$ over mean anomaly (fast variable u are considered only). For this it is needed to calculate functions $< r^{\eta} >$ and $< v^{\zeta} r^{\eta} >$, where ζ, η are integer numbers. Symbol $< >$ denotes averaging over mean anomaly. Using standard trigonometric formulas and the identities of elliptic motion we can write

$$< r^{\eta} > = \frac{1}{\tau}\int_0^{\tau} r^{\eta} dt = \frac{\alpha^{2\eta}}{2\pi}\int_0^{2\pi}\left(1-\sqrt{1-s^2}\cos u\right)^{\eta+1} du, \tag{19}$$

where

$$r\frac{du}{dt} = \frac{2\pi\alpha^2}{\tau}, \quad \tau = \frac{2\pi\alpha^3}{k} = \frac{2\pi}{n}. \tag{20}$$

Integrating (19) we obtain

$$< r^{\eta} > = s^{\eta+1}\alpha^{2\eta}P_{\eta+1}\left(\frac{1}{s}\right), \tag{21}$$

where $P_{\eta+1}\left(\frac{1}{s}\right)$ are Legendre polynomials, $s = \sqrt{1-e^2}$, $\alpha = \sqrt{a}$.

Functions $f_{ij}^{(1)}$ also contain v and r in form $v^{\zeta} r^{\eta}$, $\quad \eta \ge \zeta; \quad \zeta, \eta = 0,1,2\ldots$.

The identity

$$v^2 = -\frac{k^2}{\alpha^2} + \frac{2k^2}{r} - \frac{\alpha^2 k^2 s^2}{r^2} \tag{22}$$

can be used to eliminate all powers of v higher then the first from $f_{ij}^{(1)}$. Then it's easy to show that

$$< v r^{\eta} > = 0. \tag{23}$$

With help of our algorithm we have constructed the single-averaging Hamiltonian function $< F >$ up to five order in ε without any truncations. The generating function of the Lie transformations W has received up to four order.

The expansion of W is

$$W = \varepsilon^2 W_{02} + \varepsilon^3 W_{03} + \varepsilon^4 W_{04} + \ldots, \tag{24}$$

where ε is mean motion of disturbing body. Functions W_{0i} in (24) can be represented in the form

$$W_{0i} = \frac{L}{n^i} f_{0i}(s,c,u,g,h). \tag{25}$$

Here functions $f_{0i}(s,c,u,g,h)$ are the dimensionless quantities, Delaunay's element $L = \sqrt{\mu a}$ is action-like factor. Consequently, we have automatic development function W in powers of $m = \frac{\varepsilon}{n}$ and W has the dimension of action.

4. Conclusion

In the present paper we considered the satellite case of the space restricted three-body problem. The averaging method in the restricted three-body problem based on Lie transformations and the modified Hill's canonical variables was proposed. The efficiency of such an algorithm was discussed for the case where the solutions of the problem requires a great number of approximations. The developments for short-period perturbation was obtained in powers of m (ratio of mean motions of Sun and satellite) but in closed form with respect to eccentricity and inclination to O (m^5).

The topic of this paper has been discussed at the International astronomical meeting "Dynamics of Solar System Bodies". Tomsk, July 27 −August 1, 2008 [12].

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ИСПОЛЬЗОВАНИЕ МОДИФИЦИРОВАННЫХ ПЕРЕМЕННЫХ ХИЛЛА В МЕТОДЕ УСРЕДНЕНИЯ

Вводятся модифицированные канонические переменные Хилла: v, G, H; r, g, h, где r – длина радиус-вектора; $v = dr / dt$; $G = \sqrt{a\mu(1-e^2)}$ и $H = G \cos i$ – переменные Делоне; $g = \omega$ – аргумент перицентра; $h = \Omega$ – долгота восходящего узла. Показывается преимущество использования таких переменных в аналитических теориях, когда нежелательным является разложение выражений для возмущений по степеням эксцентриситета. В качестве примера рассматривается однократно усредненная аналитическая модель в рамках ограниченной задачи трех тел для случая движения спутника планеты.

Ключевые слова: *небесная механика, теория возмущений, метод усреднения, преобразования Ли, канонические переменные Хилла, ограниченная задача трех тел, динамика спутников планет.*

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