UDC 530.1; 539.1

ANALYTICAL PROPER ELEMENTS FOR THE SATELLITE SYSTEMS

T. S. Boronenko

Department of Theoretical Physics, Tomsk State Pedagogical University, Kievskaya str., 60, 634061 Tomsk, Russia.

E-mail: boron@tspu.edu.ru

In this paper, we present the theoretical basis of the calculation of proper elements for the irregular satellites of the giant planets. We use the averaging method for solving the restricted three-body problem. This method is based on applying transformations Lie in the space of Pfaff. Expressions for the short-period perturbations are obtained in the form of series in powers of the small parameter ϵ (the ratio of the mean motions of the Sun and satellite) and in a closed form relative eccentricities and inclinations. This is important, as the specific application objects have orbits with large values of these parameters.

Keywords: *proper elements, Lie transformation, Pfaff's space, irregular satellites.*

1 Introduction

In celestial mechanics for the description of dynamics of mechanical systems are often used orbital elements. In nonchaotic dynamical systems, the proper elements are functions of integrals of motions that depend on the initial conditions. Thus, we can calculate the proper elements from their osculating Keplerian elements. It is assumed that the irregular satellites of giant planet have weakly chaotic dynamics [1] and therefore their proper elements are practically constant for very large periods of time. However, to achieve this requires the construction of perturbation theory with a very high degree of accuracy. The most convenient in this respect is the method of Lie series mappings. Despite the seeming simplicity this method, in the process of solving practical problems require a large number of transformation of some variables on others. For example, the perturbing function, as a rule, it is very difficult, and sometimes impossible, express through canonical variables. Usually the disturbing function is expressed in terms of Keplerian elements. Therefor, it is conveniently to carry out all transformations in these elements. Using the Pfaff's method we can implement this idea.

Pfaff method is based on the use of 1-form or Pfaffian. The equations of motion of Pfaff was published in 1815. They are remarkable elegant but not very well known. These equations have already been used in celestial mechanics. The main sources are [2,3]. For the first time in the article of R. Broucke [3] the theory of changes of variables in Pfaff's equations was developed. He showed that this method allows the use of a very wide class of variables. Our goal is to find the connection between the canonical variables and some arbitrary variables (the orbital elements) of phase space and to use the Pfaff's method in the algorithm of Lie transformation. Following is brief description of the method. A more detailed description can be found in articles [4–6].

2 Description of the method

The algorithm of Lie transformation is reduced to the successive computing the Poisson brackets in canonical variables q, p. As shown in [7], invariant nature of connection between 1-form $pdq - Hdt$ and its curl lines gives you the opportunity to write the equations of motion in any $2n+1$ coordinate system in the extended phase space q, p, t. That is, you can write:

$$pdq - Hdt = X_1 dx^1 + \ldots + X_{n+1} dx^{n+1}, \tag{1}$$

where $x^{n+1} = t$ and $X_{n+1} = -H$. Then the system of the equations of motion of Pfaff can be represented as

$$(rot\boldsymbol{X})d\boldsymbol{x} = 0, \tag{2}$$

or in scalar form

$$\sum_{j=1}^{n+1} \left(\frac{\partial X_i}{\partial x^j} - \frac{\partial X_j}{\partial x^i} \right) dx^j = 0, \;\; i = 1, ..., n+1. \tag{3}$$

Pfaffian perturbed two-body problem for satellite case can be written as:

$$\Phi = \mathbf{v} \cdot d\,\mathbf{x} - Fdt = \mathbf{v} \cdot d\,\mathbf{x} - \left(\frac{\mathbf{v}^2}{2} - \frac{\mu}{r} - R \right) dt. \tag{4}$$

In (4) $\mathbf{x}$-vector position; $\mathbf{v}$ – vector of velocity of the satellite at the orbital plane; F – total energy of the system; μ is gravitational constant of Newton, multiplied by mass of the planet; R – perturbing function. Next, we will use the following vector of elliptic elements $\xi = (\alpha, \eta, \gamma, u, g, h)$, $\alpha = \sqrt{a}$, $\eta = \sqrt{1-e^2}$, $\gamma = \cos i$, $g = \omega$, $h = \Omega - \acute{\lambda}$, where a – semimagor axis, e – eccentricity, i – inclination, ω – argument of periapsis, Ω – longitude of the ascending node, n – mean motion of a satellite, $\acute{\lambda}$ – longitude of disturbing body, u – eccentric anomaly, which is connected with the time using equation

$$u - e\sin u = n(t - t_0).$$

Pfaffian in terms of Kepler's elements can be obtained if we express x and y through these elements and use equation (4). The result is:

$$\Phi = \alpha k \frac{\eta}{\sqrt{1-\eta^2}} \sin u \, d\eta + \alpha k(1 - \sqrt{1-\eta^2}\cos u) du + \alpha k\eta \, dg + \alpha k\eta \, dh - H \, dt, \quad (5)$$

where Pfaff's vector has the form:

$$\mathbf{X}(0,\, \alpha k \frac{\eta}{\sqrt{1-\eta^2}} \sin u,\, 0,\, \alpha k(1 - \sqrt{1-\eta^2}\cos u),\, \alpha k\eta,\, \alpha k\eta,\, -H). \quad (6)$$

With help of (3) and (6) we obtain the following equations of motion:

$$\begin{aligned}
\frac{d\alpha}{dt} &= \frac{\alpha^2}{kr}\frac{\partial H}{\partial u},\\
\frac{d\eta}{dt} &= -\frac{\alpha\,\eta}{kr}\frac{\partial H}{\partial u} + \frac{1}{\alpha\, k}\frac{\partial H}{\partial g},\\
\frac{d\gamma}{dt} &= -\frac{\gamma}{\alpha k\eta}\frac{\partial H}{\partial g} + \frac{1}{\alpha\, k\eta}\frac{\partial H}{\partial h},\\
\frac{du}{dt} &= -\frac{\alpha^2}{kr}\frac{\partial H}{\partial \alpha} + \frac{\alpha\,\eta}{kr}\frac{\partial H}{\partial \eta} - \frac{\alpha\eta\sin u}{k\sqrt{1-\eta^2}r}\frac{\partial H}{\partial g},\\
\frac{dg}{dt} &= -\frac{1}{\alpha\, k}\frac{\partial H}{\partial \eta} + \frac{\alpha\eta\sin u}{k\sqrt{1-\eta^2}r}\frac{\partial H}{\partial u} + \frac{\gamma}{\alpha k\eta}\frac{\partial H}{\partial \eta},\\
\frac{dh}{dt} &= -\frac{1}{\alpha k\eta}\frac{\partial H}{\partial H}, \qquad (7)
\end{aligned}$$

where $r = \alpha^2(1 - e\sin u)$, $k = \sqrt{\mu}$. Hamiltonian $H = H_0(\alpha) + \epsilon R(\alpha, \eta, \gamma, u, g, h)$, and ϵ is a small parameter. Right parts of equations (5) define the structure of Poisson brackets, which are the basis of this method.

3 Conclusion

For a long time proper elements have been used for the sole purpose of identifying the asteroid families. In [1] is described the history of the use of this method to the study of the dynamics of the satellites of giant planets. The list of problems to be solved by applying the proper elements is long. For example, it is stable chaos, problems of resonance, determining the age of objects and so on.

To obtain the analytical expressions for proper elements, the method set forth above was implemented using the computer system Mathematica. Expressions for the short-period perturbations were obtained in the form of series in powers of the small parameter ϵ (the ratio of average movements of the Sun and satellite) to O(6). Unlike other works, these expansions were obtained in a closed form relative eccentricities and inclinations. This is important because the irregular satellites of the giant planets have orbits with large values of these parameters. In results, analytical expressions for the proper element of a^* was obtained with high accuracy. Comparison with the results of the work [1] gave a good match. The next stage of our research is to determine analytical expressions for the elements of e_p and I_p and further study the global location of secular resonances in the space of proper elements.

Acknowledgement

Subject of this article was discussed at the VIII Russian scientific conference "Fundamental and applied problems modern mechanics section "Flight dynamics and celestial mechanics".Tomsk, 22–26 April 2013.

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Received 09.11.2014

Т. С. Бороненко

АНАЛИТИЧЕСКИЕ СОБСТВЕННЫЕ ЭЛЕМЕНТЫ ДЛЯ СПУТНИКОВЫХ СИСТЕМ

В статье дается метод вычисления собственных элементов для нерегулярных спутников планет-гигантов. Метод основан на использовании преобразований Ли в пространстве Пфаффа в рамках ограниченной задачи трех тел. Получены разложения для короткопериодических возмущений по степеням малого параметра ϵ (отношение средних движений Солнца и спутника). Разложения по степеням эксцентриситета и наклонения орбиты спутника не производятся. Это важно, так как объекты приложения имеют орбиты с большими величинами этих параметров.

Ключевые слова: *собственные элементы, преобразования Ли, пространство Пфаффа, нерегулярные спутники.*

Бороненко Т. С., кандидат физико-математических наук, доцент.
Томский государственный педагогический университет.
Ул. Киевская, 60, 634061 Томск, Россия.
E-mail: boron@tspu.edu.ru