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New States of Gauge Theories on a circle

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We study a one-dimensional large-N $U(N)$ gauge theory on a circle as a toy model of higher dimensional Yang-Mills theories. We find a new class of saddle point solutions in this theory. These solutions are characterized by the expectation values of the Polyakov loop operators, which wind the circle different times. We find two evidences that these solutions appear as intermediate states in certain dynamical processes. One is from a numerical calculation and another is from the dual gravity. The similar solutions exist in a wide class of $SU(N)$ and $U(N)$ gauge theories on S^1 including QCD and pure Yang-Mills theories in various dimensions if $N \geq 3$.

Keywords: *gauge theories, Yang-Mills theory, gravity.*

Introduction. We consider gauge theories on a circle. Here we have two different classes of the theories depending on the choices of the circle: temporal circle in Euclidian theory or spatial circle. In the case of the temporal circle, the theories describe the finite temperature systems [1]. In the case of the spatial circle, these theories would be regarded as the extra-dimension models of our world [2–4], which may naturally appear in string theory. Hence both cases are significant in theoretical physics.

In this study [5], we find a new class of saddle point solutions in the $U(N)$ and $SU(N)$ gauge theories on a circle in the case $N \geq 3$. These solutions exist in both temporal and spatial circle cases, and may be relevant in the above studies. We investigate their properties through both analytical and numerical methods. Especially we find several evidences that these solutions might play important roles in certain dynamical processes.

Although these new saddle point solutions exist in various gauge theories on a circle, in order to understand them concretely, we mainly consider the following large-N BFSS type matrix model at finite temperature,

$$S = \int_0^\beta dt \mathrm{Tr} \left(\sum_{I=1}^{D} \frac{1}{2} \left(D_t Y^I \right)^2 - \sum_{I,J} \frac{g^2}{4} [Y^I, Y^J]^2 \right). \tag{1}$$

Here Y^I is adjoint scalar ($I = 1, \cdots, D$) and $D_t = \partial_t - i[A_t, \cdot]$. β and A_t denote the inverse temperature $1/T$ and the gauge field on the Euclidean time circle, respectively.

We consider this model for the following reasons. This model is obtained from a dimensional reduction of a higher dimensional pure Yang-Mills theory and inherits several properties. For example, an analog of the confinement/deconfinment transition happens at large N. Besides non-perturbative analyses of this model have been developed well via numerical method [6] and analytic method by the $1/D$ expansion [7]. Hence this model is a good toy model of higher dimensional gauge theories. In addition, this model is also obtained as an effective theory of N D branes in the case $D = 9$. So we can study the relation to the gravity dual too [6].

In the case of the spatial S^1 circle, we will argue what the gravity duals of our new saddle point solutions are. Especially we will see that these gravity solutions play important roles in a real-time process of the Gregory-Laflamme transition [8–10]

The Phase Structure. We briefly review the phase structure of the model (1) studied in [7]. The phases of the gauge theories are characterized by the expectation value of the Polyakov loop operator

$$P \equiv \frac{1}{N} \mathrm{TrP} \exp \left(i \int_0^\beta A_t dt \right) = \frac{1}{N} \sum_{k=1}^{N} e^{i\alpha_k}, \tag{2}$$

where we have taken the static diagonal gauge:

$$(A_t)_{ij} = \alpha_i \delta_{ij} / \beta, \quad (i, j = 1, \cdots, N). \tag{3}$$

Here $\partial_t \alpha_i = 0$ is imposed and α_i has a periodicity $\alpha_i = \alpha_i + 2\pi$. The phase is confined if $\langle |P| \rangle = 0$, and is deconfined if $\langle |P| \rangle \neq 0$. The eigenvalue distribution function

$$\rho(\alpha) \equiv \frac{1}{N} \sum_{i=1}^{N} \delta(\alpha - \alpha_i) \tag{4}$$

is also a convenient tool to describe the phases.

In the gauge (3), the configuration of the eigenvalue $\{\alpha_k\}$ determines the value of the Polyakov loop operator (2). Since $\{\alpha_k\}$ can be regarded as the positions of N particles on S^1, this N particle problem governs the phase structures. Then we can interpret the mechanism of the confinement and deconfinement as follows. If repulsive forces between the eigenvalue $\{\alpha_k\}$ are dominant, the eigenvalues tend to spread on the S^1 and the stable configuration is

$$\alpha_k = 2\pi k/N + c \pmod{2\pi}, \tag{5}$$

and their permutations. Here c is a k-independent constant, which is discretized as $2\pi n/N$ ($n \in Z_N$) in the $SU(N)$ case. Then $|P| = 0$ is satisfied and the confinement phase is realized. In this case, the eigenvalue distribution $\rho(\alpha)$ is uniform as in Fig. 1 (I).

Oppositely, if attractive forces are dominant, the eigenvalues tend to clump and the configuration

$$\alpha_k = c \pmod{2\pi} \tag{6}$$

would be stable. Usually quantum effects disturb this configuration and the eigenvalue distribution is smeared around c as shown in Fig. 1 (II) or (III). At this time, $|P| \neq 0$ is satisfied, and the deconfinement phase is realized. Therefore the phases in the gauge theories may be related to the forces between the eigenvalue $\{\alpha_k\}$.

The authors in [7] analyzed the model (1) by using the $1/D$ expansion, and explicitly confirmed the relation between the forces and the phases. They also revealed the phase structure. The confinement phase is stable at low temperature, and at $T_{c1} = (D\lambda)^{1/3}/\log D$, the confinement/deconfinement transition happens, and, above it, the deconfinement phase is stable. (A Gross-Witten-Wadia (GWW) type transition [11, 12] also occurs at T_{c2} near T_{c1}.) See Fig. 3. Correspondingly the repulsive and the attractive forces are dominant at each temperature.

Multi Peak Saddle Point Solutions. Once we understand the relation between the phases and the forces between $\{\alpha_i\}$, we intuitively notice that, if the attractive forces are dominant at high temperature ($T \geq T_{c1}$), the configurations with multiple mobs of the eigenvalues may be possible as a solution of the model, if the attractive forces between the mobs are balanced.

Indeed we can confirm that these solutions exist in the model (1) via the $1/D$ expansion [5]. These solutions are unstable, since the mobs are attracted each other, and they are saddle points. We call them "multi-peak solution" or "multi-cut solution". Particularly, the construction of the solution $\rho(\alpha)$ with a Z_m symmetry $\alpha \to \alpha + 2\pi/m$ is very simple. See $m = 2$ case in Fig. 2. We call them Z_m solution and find the critical temperatures for these solutions as shown in Fig. 3.

Further, we evaluate observables of Z_m solutions and compare them with a Monte Carlo simulation. Then we find quantitative agreements. For details, see the original paper [5].

Stochastic Time Evolution. We will study the profile of the effective potential for $\{\alpha_i\}$. Such a profile is crucial to investigate the decay process of an unstable state to a stable state. Although our effective potential is thermal and is not directly related to the real-time decay process, if the decay happens sufficiently slowly or adiabatically, the decay process may reflect the profile of the potential.

Of course we cannot draw the potential explicitly, since we have infinite variable α_i at large N. However we possibly read off the relevant part of the profile of the potential through the following procedure. Suppose that we can integrate out adjoint scalar Y^I in the model (1) and obtain an effective action for the gauge field $S_{\text{eff}}(\alpha_i)$. Then we consider an unstable solution in the action $S_{\text{eff}}(\alpha_i)$. We smoothly deform $\{\alpha_i\}$ from this solution such that $S_{\text{eff}}(\alpha_i)$ is becoming smaller. By repeating this deformation, $\{\alpha_i\}$ may settle down to a stable configuration finally. Then we can speculate the profile of the potential between the unstable solution and the stable configuration from the history of the deformation of $\{\alpha_i\}$.

We investigate this process by using a stochastic time evolution of a Monte Carlo calculation, which is designed such that the above process is realized. We assign a discrete "stochastic time" s for α_i and Y^I, which is distinguished from the Euclidean time t in the model (1). We take an appropriate initial configuration at $s = 0$, and, to gain the time s, we update $\alpha_i(s)$ and $Y^I(s)$ through the following rule:

1. Set a trial configuration $\alpha_{i,\text{trial}}(s+1) = \alpha_i(s) + r$, where r is a small random number.
2. If $S[\alpha_{i,\text{trial}}(s+1)] \leq S[\alpha_i(s)]$, we accept this trial configuration as $\alpha_i(s+1) = \alpha_{i,\text{trial}}(s+1)$.
3. Even if $S[\alpha_{i,\text{trial}}(s+1)] > S[\alpha_i(s)]$, we still accept $\alpha_{i,\text{trial}}(s+1)$ with the probability $\exp(-S[\alpha_{i,\text{trial}}(s+1)] + S[\alpha_i(s)])$, and, if it is rejected, we retain $\alpha_i(s+1) = \alpha_i(s)$.
4. Update the scalar field $Y^I(s)$ many times such that they arrive at an equilibrium for the given configuration $\{\alpha_i(s+1)\}$, and use this state as $Y^I(s+1)$.

The last step might correspond to the path integral of Y^I and is taken to focus on the dynamics of $\{\alpha_i\}$ in $S_{\text{eff}}(\alpha_i)$. Through this evolution, $\{\alpha_i\}$ is deformed gradually such that the action $S_{\text{eff}}(\alpha_i)$ tends to be smaller as we intended. Note that this procedure for $\{\alpha_i\}$ is based on the Metropolis algorithm.

Then we will see that this method captures several characteristic properties of the model (1).

Decay Patterns in the Stochastic Time Evolution. We investigate the stochastic time evolution of the unstable states of the model (1). At $T > T_{c1}$, we take the unstable uniform solution (5) as the initial state, and evaluate the evolutions repeatedly by changing the temperature and random number. Then we observe the following two evolution patterns depending on temperature [1].

Direct Decay. In this pattern, the unstable uniform solution directly evolves to a more stable one-peak state (deconfinement configuration). This pattern is mainly observed at lower temperature $T_{c1} < T < c(D)T_{c1}$. Here $c(D)$ is a constant, which seems to depend on D as $c(D = 2) \sim 3.5$ and $c(D = 9) \sim 5.0$, although the change of the decay pattern at $c(D)T_{c1}$ is not sharp.

Cascade Decay. In this pattern, the unstable uniform solution first evolves to a multi-peak state. Then the peaks attract each other, and two of them collide and merge into one peak. By repeating such collisions, the number of the peaks decreases one by one, and it finally achieves the one-peak state. This pattern is dominant at high temperature $c(D)T_{c1} < T$. The m-peak states with a larger m tend to appear at higher temperature. An example is shown in Fig.4.

In this way, the multi-peak states appear as the intermediate states in the stochastic decay process of the unstable uniform state. These results indicate that the multi-peak states lie between the uniform and the one-peak configuration in the potential valley of $S_{\text{eff}}(\alpha_i)$. We will see the relevance of this profile of the potential in the dual gravity theory in the next paragraph.

Gravity Duals of the Multi-cut Solutions. Now we consider the gauge/gravity correspondence and discuss what the dual gravity solutions of the multi-cut solutions in the model (1) with $D = 9$ are.

To consider it, we regard the temporal circle of (1) as a spatial one. We rename β as L, and call the circle S^1_L. Then it has been shown that the model (1) is obtained as an effective theory of N D0 branes at high temperature by taking a T-dual along the S^1_L [6]. We call the dual circle $S^1_{1/L}$ and define the radius of $S^1_{1/L}$ as L' ($\equiv 1/L$). In this model, N D0 branes are localized on $S^1_{L'}$, and the eigenvalue $\{\alpha_i\}$ in the model (1) now represents the position of the branes on $S^1_{L'}$.

Through the gauge/gravity correspondence, the IIA supergravity gravity with $S^1_{L'}$ would describe this system in the strong coupling. The gravity duals of the uniform and one-cut solutions have been predicted [6]. The dual of the uniform solution would be the uniformly smeared black D0 brane solution whose horizon winds the $S^1_{L'}$. This geometry is a kind of a black string. The dual of the one-cut solution would be the black D0 brane solution whose horizon is localized on the $S^1_{L'}$. These correspondences seem reasonable through the relation between the eigenvalue distribution of $\{\alpha_i\}$ and the D0 brane distribution.

It has been revealed that their stabilities are also consistent. The smeared black D0 brane is stable at small L', which means large L, and unstable at large L' due to the Gregory-Laflamme instability. This is similar to the uniform solution which is stable only at large L. Oppositely, the localized black D0 brane is stable at large L' and it agrees with the one-cut solution.

Now we consider the dual geometry of the m-cut solution in the gauge theory. According to the above correspondences, the m-cut solution would correspond to m black D0 branes localized on $S^1_{L'}$. Such multi black branes have been known as the solutions of the supergravity, and are unstable due to the gravitational attractive forces between the branes. It is consistent with the instability of multi-cut solutions in the gauge theory. Indeed the existence of the dual phases in the gauge theories corresponding to these multi black brane solutions in the Kaluza-Klein gravities has been predicted in Ref. [13]. The saddle point solutions in our article provide evidence for this conjecture.

Such multi black holes (connected by thin black strings) appear in a real-time decay process of a black string [9, 10, 14]. As we mentioned, the black string is unstable at large L', and a localized black hole is thermodynamically favored. The authors in Refs. [9,10] examine the decay process of a meta-stable black string by solving the Einstein equation numerically. They found that several points of the horizon are growing and other parts are shrinking owing to the Gregory-Laflamme instability. As a result, a sequence of black holes joined by thin black string segments appear along $S^1_{L'}$. See Fig. 5. However the numerical analysis did not work for sufficiently long time and the final state of this process has not been found. Since this configuration is unstable, it might evolve to a single localized black hole joined by an extremely thin black string.

Remarkably this time evolution of the black string is similar to the stochastic time evolution of the unstable uniform configuration in the gauge theory. The multi-peak state appears as the intermediate state in both cases. This agreement may imply that the dynamical stability in the gravity may be related to the thermodynamical stability in the gauge theory (Recall that the stochastic time evolution may reflect the profile of the thermodynamical effective potential of the gauge theory.)

[1]The movies for the stochastic evolutions are available on `http://www2.yukawa.kyoto-u.ac.jp/~azuma/multi_cut/index.html`.

Summary. We have found the multi-peak saddle point solutions in the model (1). The important condition for the existence of these solutions is the attractive forces between the eigenvalues, which typically realize in case the deconfinement phase is stable. Since the deconfinement phase generally exist in various gauge theories, we expect that similar multi-peak solutions also exist there. Indeed we can confirm it in the four dimensional pure Yang-Mills theory and QCD [5]. These solutions exist even in finite N case as well if $N \geq 3$.

Through the stochastic time evolution of the one-dimensional gauge theory (1), we found that these multi-peak states may exist as the intermediate states between the unstable confinement phase and the stable deconfinement phase at high temperature. This result indicates that, even in the real-time decay process of the unstable confinement phase, some related intermediate states might appear. It is valuable to investigate it further but one difficulty is the definition of the multi-peak states in the real-time formalism, since the theory does not have the temporal circle anymore.

A natural generalization of this study is the stochastic evolution of an unstable confinement phase in higher dimensional gauge theories. We can also take the S^1 as the spatial circle as we considered. Then we do not have any difficulty about the definition of the multi-peak solutions in the real-time formalism. One interesting application is to the phenomenology of particle physics. In some models, the gauge field of the spatial S^1, which is an extra-dimension of our world, is identified as the Higgs field [2–4]. If we consider the time evolution of such models in early universe, certain multi-peak configurations might appear and probably play some roles.

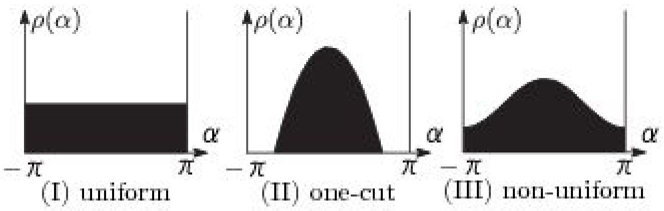

Figure 1: The distribution function $\rho(\alpha)$ in three states.

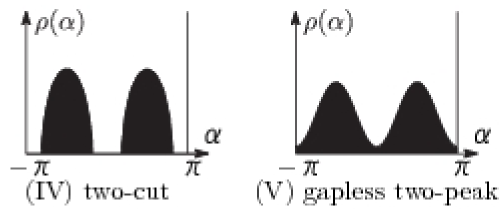

Figure 2: The distribution function $\rho(\alpha)$ for two-peak states with two gaps (=two cuts) (IV) and gapless (V).

We have also studied the application to the gauge/gravity correspondence by identifying the eigenvalue distribution of the spatial gauge field and the positions of the D branes on the spatial circle in the Kaluza-Klein gravity through the T-duality. There the multi-peak solutions in the gauge theory may correspond to the multi black branes localized on the circle. In this case, we found an interesting similarity between the decay process of the black string [9, 10, 14] and the stochastic time evolution in the gauge theory. It must be valuable to evaluate the time evolution of the gauge theories on a spatial circle to see the similar behaviours directly. Finally we have one missing object. The gravity duals of the multi-cut solutions along the temporal circle in supersymmetric Yang-Mills theories have not been found. Although the gravity dual of the temporal multi-cut solutions in $(p+1)$-dimensional pure Yang-Mills theory would be obtained through the T-dual and the double Wick rotation of the multi black black p branes localized on the temporal circle according to the conjecture discussed in Ref. [15], the gravity duals of the supersymmetric Yang-Mills theories might be given by different generalizations of a black p branes. If we can find such brane solutions, it might be important in the context of the uniqueness theorem in general relativity.

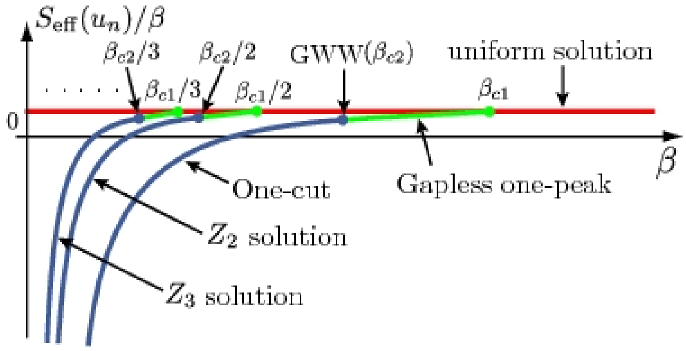

Figure 3: A sketch of the effective action of the several solutions in the model (1). The green and blue lines are for the gapless and the gapped solutions respectively. Note that this sketch is not rigidly plotted based on the equations, and the GWW point β_{c2} is much close to β_{c1} in the actual plot at large D.

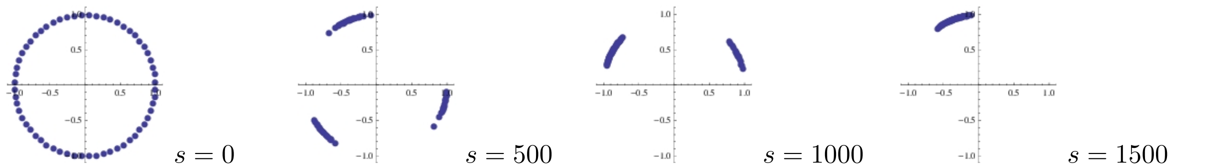

Figure 4: The stochastic evolution of the eigenvalue distribution for $D = 9, N = 60, T = 6.7T_{c1}$. $e^{i\alpha_k}$ $(k = 1, \cdots, N)$ are plotted.

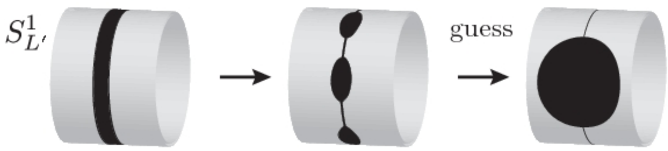

Figure 5: A sketch of the decay process of the meta-stable black string in Refs. [9, 10]. The last step is just a speculation. The inside of the apparent horizon is depicted as the black region. In Ref. [10], an interesting fractal structure, which is akin to a low viscosity fluid, is observed but we omit it in this sketch.

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НОВЫЕ СОСТОЯНИЯ КАЛИБРОВОЧНЫХ ТЕОРИЙ НА ОКРУЖНОСТИ

Исследована одномерная $U(N)$ теория на окружности для большого N в качестве модели теорий Янга-Миллса в пространствах высоких размерностей. Найден новый класс решений для седловой точки. Найдено два подтверждения того, что эти решения возникают как промежуточные состояния в определенных динамических процессах. Аналогичные решения существуют в широком классе калибровочных теорий $SU(N)$ и $U(N)$, включая КХД и чистые теории Янга-Миллса в различных измерениях, если $N \geq 3$

Ключевые слова: *калибровочные теории; теория Янга-Миллса; гравитация*

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