UDC 530.1; 539.1

COSMOLOGICAL PERTURBATIONS IN THE THEORY OF GRAVITY WITH NONMINIMAL KINETIC COUPLING

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We consider cosmological perturbations in the theory of gravity with nonminimal kinetic coupling. The Lagrangian of the theory contains the term $\eta G^{ij}\phi_{,i}\phi_{,j}$, and represents the particular example of a general Horndeski Lagrangian, which results in second-order field equations. We derive a complete set of equations for scalar, vector and tensor pertubations. The tensor modes are analyzed in detail. It is shown that their behavior inside the Hubble horizon differs cardinally from the analogous behavior of tensor modes in Friedmann cosmology.

Keywords: *nonminimal kinetic coupling, cosmological perturbations.*

1 Introduction

Natural modifications of general relativity are the models describing possible nonminimal coupling between matter fields and the curvature. The most intensively and widely studied are various nonminimal generalizations of scalar-tensor theories of gravity which have numerous cosmological applications (see Ref. [1] for detailed reviews of these and other models). An especial approach to modified theories of gravity represent models allowing for nonminimal coupling between derivatives of dynamic quantities of matter fields and the curvature. The most general scalar-tensor theory of such type was suggested in the 70-es of the last century in the Horndeski work [2]. Horndeski developed his theory on the base of mathematical facts but later the same results were obtained on the basis of more intuitive approach from Galileons research [3].

The simplest Lagrangian in the Horndeski theory contains a term $G^{\mu\nu}\phi_{,\mu}\phi_{,\nu}$ providing nonminimal kinetic coupling of a scalar field to the curvature. Cosmological applications of such theory have been intensively investigated in [4, 5]. In particular, in our recent works [4] we have found that the non-minimal derivative coupling provides an essentially new inflationary mechanism and naturally describes transitions between various cosmological phases without any fine-tuning potential.

It is worth noticing that most of works on cosmologies with nonminimal kinetic coupling had focused on the background cosmological evolution. However, in order to reveal the full structure and the physical implications of the theory, one must proceed to the detailed investigation of the perturbations. The linear scalar perturbations was discussed in Ref. [6]. The aim of this work is to derive the complete set of equations for scalar, vector and tensor cosmological pertubations in the theory of gravity with nonminimal kinetic coupling.

2 Field equations

The action of the theory of gravity with nonminimal kinetic coupling is given as follows

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{8\pi} - \left(\varepsilon g_{\mu\nu} + \eta G_{\mu\nu}\right)\phi^{,\mu}\phi^{,\nu} - 2V(\phi)\right] + S_m, \quad (1)$$

where S_m is the action for ordinary matter (not including the scalar field), and η is the coupling parameter with dimension of $(length)^2$. Varying the action with respect to $g_{\mu\nu}$ and ϕ gives the field equations, respectively:

$$G_{\mu\nu} = 8\pi\left[T^{(m)}_{\mu\nu} + T^{(\phi)}_{\mu\nu} + \eta\Theta_{\mu\nu}\right], \quad (2)$$

$$[\varepsilon g^{\mu\nu} + \eta G^{\mu\nu}]\nabla_\mu\nabla_\nu\phi = -V_\phi, \quad (3)$$

where $V_\phi \equiv dV(\phi)/d\phi$, $T^{(m)}_{\mu\nu}$ is a stress-energy tensor of ordinary matter, and

$$\begin{aligned} T^{(\phi)}_{\mu\nu} &= \varepsilon[\nabla_\mu\phi\nabla_\nu\phi - \tfrac{1}{2}g_{\mu\nu}(\nabla\phi)^2] - g_{\mu\nu}V(\phi), \quad (4)\\ \Theta_{\mu\nu} &= -\tfrac{1}{2}\nabla_\mu\phi\,\nabla_\nu\phi\,R + 2\nabla_\alpha\phi\,\nabla_{(\mu}\phi R^\alpha_{\nu)} \\ &+ \nabla^\alpha\phi\,\nabla^\beta\phi\,R_{\mu\alpha\nu\beta} + \nabla_\mu\nabla^\alpha\phi\,\nabla_\nu\nabla_\alpha\phi \\ &- \nabla_\mu\nabla_\nu\phi\,\Box\phi - \tfrac{1}{2}(\nabla\phi)^2 G_{\mu\nu} \\ &+ g_{\mu\nu}\big[-\tfrac{1}{2}\nabla^\alpha\nabla^\beta\phi\,\nabla_\alpha\nabla_\beta\phi \\ &+ \tfrac{1}{2}(\Box\phi)^2 - \nabla_\alpha\phi\,\nabla_\beta\phi\,R^{\alpha\beta}\big]. \quad (5)\end{aligned}$$

Now let us consider a spatially-flat FRW cosmological model with the metric

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j, \quad (6)$$

where $a(t)$ is the scale factor, and $H(t) = \dot{a}(t)/a(t)$ is the Hubble parameter. Supposing homogeneity and isotropy, we also get $\phi = \phi(t)$ and $T^{(m)}_{\mu\nu} =$

diag(ρ, p, p, p), where $\rho = \rho(t)$ is the energy density and $p = p(t)$ is the pressure of matter.

The general field equations (2), (3) written for the metric (6) yield

$$3H^2 = 8\pi\rho + 4\pi\dot{\phi}^2 \left(\varepsilon - 9\eta H^2\right) + 8\pi V(\phi), \tag{7}$$

$$2\dot{H} + 3H^2 = -8\pi p + 8\pi V(\phi)$$
$$- 4\pi\dot{\phi}^2 \left[\varepsilon + \eta\left(2\dot{H} + 3H^2 + 4H\ddot{\phi}\dot{\phi}^{-1}\right)\right], \tag{8}$$

$$\varepsilon(\ddot{\phi} + 3H\dot{\phi}) - 3\eta(H^2\ddot{\phi} + 2H\dot{H}\dot{\phi} + 3H^3\dot{\phi}) = -V_\phi, \tag{9}$$

where a dot denotes derivatives with respect to time.

The most intriguing feature of this model is existence of an essentially new inflationary mechanism which does not depend on a form of scalar potential. The inflation is driven by terms in the field equations responsible for the nonminimal derivative coupling. At early times these terms are dominating, and the cosmological evolution has the quasi-de Sitter character $a(t) \propto e^{H_\eta t}$ with $H_\eta = 1/\sqrt{9\eta}$. Later, in the course of the cosmological evolution the domination of η-terms is canceled, the usual matter comes into play, and the Universe enters into the matter-dominated epoch.

3 Perturbations

Let us consider the perturbed FRW metric:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \tag{10}$$

where $\bar{g}_{\mu\nu}$ is the background metric:

$$\bar{g}_{00} = -1,\ \bar{g}_{i0} = \bar{g}_{0i} = 0,\ \bar{g}_{ij} = a^2(t)\,\delta_{ij}$$

and $h_{\mu\nu} = h_{\nu\mu}$ are small perturbations. Hereafter, we perform the calculations supposing that $h_{i0} = h_{0i} = 0$ and using the Newtonian gauge for scalar perturbations. Generally, we have

$$h_{00} = -E, \tag{11}$$
$$h_{i0} = a\left(\partial_i F + G_i\right), \tag{12}$$
$$h_{ij} = a^2\left(A\delta_{ij} + \partial_{ij}B + \partial_i C_j + \partial_j C_i + D_{ij}\right), \tag{13}$$

where A, B, E, F are scalars (helicity 0), C_i, G_i are transverse vectors (helicity 1), and $D_{ij} = D_{ji}$ is a transverse traceless tensor (helicity 2). All functions depend on $\mathbf{x}$ and t and obey the following conditions:

$$\partial_i C_i = \partial_i G_i = 0, \quad \partial_i D_{ij} = 0, \quad D_{ii} = 0,$$

Also we have the following perturbations of the stress-energy tensor:

$$\delta T_{00} = -\rho h_{00} + \delta\rho, \tag{14}$$
$$\delta T_{i0} = p h_{i0} - (\rho + p)\delta u_i, \tag{15}$$
$$\delta T_{ij} = p h_{ij} + a^2\delta_{ij}\delta p, \tag{16}$$

and the scalar field:

$$\phi(\mathbf{x}, t) = \bar{\phi}_0(t) + \delta\phi(\mathbf{x}, t). \tag{17}$$

Substituting the pertubed values (11)-(17) into the field equations (2) and taking into account the background equations (7)-(9), one can obtain the following equations for perturbations:

00-*component*:

$$P_1 E + P_2\dot{A} + P_3\nabla^2 A + P_4\delta\dot{\phi} + P_5\nabla^2\delta\phi + P_6\delta\rho = 0. \tag{18}$$

ii-component:

$$Q_1 E + Q_2\dot{E} + Q_3\nabla^2 E + Q_4\dot{A} + Q_5\ddot{A} + Q_6\nabla^2 A + Q_7\delta\dot{\phi} + Q_8\delta\ddot{\phi} + Q_9\nabla^2\delta\phi + Q_{10}\delta p = 0. \tag{19}$$

0*i-component*:

$$R_1\partial_i E + R_2\partial_i\dot{A} + R_3\nabla^2\dot{C}_i + R_4\partial_i\delta\phi + R_5\partial_i\delta\dot{\phi} + R_6\delta u_i = 0. \tag{20}$$

ij-component:

$$S_1\partial_i\partial_j E + S_2\partial_i\partial_j A + S_3\partial_i\partial_j\delta\phi + S_4\left(\partial_i\dot{C}_j + \partial_j\dot{C}_i\right) + S_5\left(\partial_i\ddot{C}_j + \partial_j\ddot{C}_i\right) + S_6\dot{D}_{ij} + S_7\ddot{D}_{ij} + S_8\nabla^2 D_{ij} = 0. \tag{21}$$

4 Tensor modes

Let us discuss tensor modes D_{ij} of cosmological perturbations. Using Eqs. (21) and (31), we obtain

$$(1 + 4\pi\eta\dot{\phi}^2)\ddot{D}_{ij} + \left(3H + 4\pi\eta(2\dot{\phi}\ddot{\phi} + 3H\dot{\phi}^2)\right)\dot{D}_{ij} - \frac{1}{a^2}(1 - 4\pi\eta\dot{\phi}^2)\Delta D_{ij} = 0. \tag{22}$$

As usually, an arbitrary transversal traceless tensor D_{ij} can be represented as a linear combination of two basic tensors $e_{ij}^{(+)}$ and $e_{ij}^{(\times)}$ with helicities $+2$ and -2, respectively, so that

$$D_{ij} = \sum_{A=+,\times} e_{ij}^{(A)}\theta^{(A)}, \tag{23}$$

where $\theta^{(A)}$ are amplitudes. Applying the Fourier transform

$$\theta^{(A)}(t, \mathbf{x}) = \int d\mathbf{k} e^{i\mathbf{k}\mathbf{x}}\theta^{(A)}(t, \mathbf{k}), \tag{24}$$

we can rewrite Eq. (22) as follows:

$$(1 + 4\pi\eta\dot{\phi}^2)\ddot{\theta} + \left(3H + 4\pi\eta(2\dot{\phi}\ddot{\phi} + 3H\dot{\phi}^2)\right)\dot{\theta} + \frac{k^2}{a^2}(1 - 4\pi\eta\dot{\phi}^2)\theta = 0. \tag{25}$$

(Hereafter, for simplicity, we have omitted the index A.) Now, it is worth considering two limiting cases.

A. $4\pi\eta\dot{\phi}^2 \ll 1$.

In this case we can neglect terms responsible for the nonminimal kinetic coupling, and Eq. (25) for tensor amplitudes $\theta^{(A)}$ reads

$$\ddot{\theta} + 3H\dot{\theta} + \frac{k^2}{a^2}\theta = 0. \tag{26}$$

This is nothing but the well-known equation describing an evolution of tensor modes in Friedmann cosmology. In case $k/a \ll H$ (outside the Hubble horizon) θ is constant; in case $k/a \gg H$ (inside the Hubble horizon) the amplitude θ behaves as a damping oscillating mode.

B. $4\pi\eta\dot{\phi}^2 \gg 1$.

In this case terms in the field equations responsible for the nonminimal kinetic coupling are dominating, and the background cosmological evolution has the quasi-de Sitter character $a(t) \propto e^{H_\eta t}$ with $H_\eta = 1/\sqrt{9\eta}$, and the scalar field behaves as $\phi(t) \propto e^{-3H_\eta t}$ (see Ref. [4]). Now, Eq. (25) takes the following form:

$$\ddot{\theta} - 3H_\eta\dot{\theta} - \frac{k^2}{a^2}\theta = 0. \tag{27}$$

As in the previous case, we can see that amplitudes of tensor modes outside the Hubble horizon, i.e., $k/a \ll H_\eta$, are constant. However, a behavior of modes inside the Hubble horizon, when $k/a \gg H_\eta$, is cardinally changed. From Eq. (27) we find $\theta \propto \exp(k\int \frac{dt}{a})$.

5 Summary

In this paper we have derived a complete set of equations for scalar, vector and tensor cosmological pertubations in the theory of gravity with nonminimal kinetic coupling. Tensor modes have been analyzed in more details. It has been shown that their behavior inside the Hubble horizon differs cardinally from the analogous behavior of tensor modes in Friedmann cosmology.

Acknowledgement

The work was supported by the Russian Government Program of Competitive Growth of Kazan Federal University and by the Russian Foundation for Basic Research Grant No. 14-02-00598.

Appendix: Coefficients of perturbed equations

In this section we present explicit expressions for coefficients P_i, Q_i, R_i, and S_i, which appear in Eqs. (18)– (21) for cosmological perturbations.

Coefficients P_i:

$$P_1 = -8\pi\left(\rho + \frac{9}{2}\eta H^2\dot{\phi}^2\right),$$
$$P_2 = 3H\left(1 + 12\pi\eta\dot{\phi}^2\right),$$
$$P_3 = -\frac{1}{a^2}\left(1 + 4\pi\eta\dot{\phi}^2\right),$$
$$P_4 = -8\pi\dot{\phi}\left(\varepsilon - 9\eta H^2\right),$$
$$P_5 = -\frac{16\pi\eta H\dot{\phi}}{a^2},$$
$$P_6 = -8\pi. \tag{28}$$

Coefficients Q_i:

$$Q_1 = 4\pi\varepsilon\dot{\phi}^2 + 32\pi\eta H\dot{\phi}\ddot{\phi} + \left(3H^2 + 2\dot{H}\right)\left(1 + 8\pi\eta\dot{\phi}^2\right),$$
$$Q_2 = H\left(1 + 12\pi\eta\dot{\phi}^2\right),$$
$$Q_3 = \frac{1}{3a^2}\left(1 + 4\pi\eta\dot{\phi}^2\right),$$
$$Q_4 = -3H\left(1 + 4\pi\eta\dot{\phi}^2\right) - 8\pi\eta H\dot{\phi}\ddot{\phi},$$
$$Q_5 = -\left(1 + 4\pi\eta\dot{\phi}^2\right),$$
$$Q_6 = \frac{1}{3a^2}\left(1 - 4\pi\eta\dot{\phi}^2\right),$$
$$Q_7 = -8\pi\left[\left(\varepsilon + 3\eta H^2 + 2\eta\dot{H}\right)\dot{\phi} + 2\eta H\ddot{\phi}\right],$$
$$Q_8 = -16\pi\eta H\dot{\phi},$$
$$Q_9 = \frac{16\pi\eta}{3a^2}\left(\ddot{\phi} + H\dot{\phi}\right),$$
$$Q_{10} = -8\pi. \tag{29}$$

Coefficients R_i:

$$R_1 = H\left(1 + 12\pi\eta\dot{\phi}^2\right),$$
$$R_2 = -\left(1 + 4\pi\eta\dot{\phi}^2\right),$$
$$R_3 = \frac{1}{2}\left(1 + 4\pi\eta\dot{\phi}^2\right),$$
$$R_4 = -8\pi\dot{\phi}\left(\varepsilon - 3\eta H^2\right),$$
$$R_5 = -16\pi\eta H\dot{\phi},$$
$$R_6 = 8\pi\left(\rho + p\right). \tag{30}$$

Coefficients S_i:

$$S_1 = -\frac{1 + 4\pi\eta\dot{\phi}^2}{2a^2},$$
$$S_2 = -\frac{1 - 4\pi\eta\dot{\phi}^2}{2a^2},$$
$$S_3 = -\frac{8\pi\eta}{a^2}\left(\ddot{\phi} + H\dot{\phi}\right),$$
$$S_4 = 4\pi\eta\dot{\phi}\ddot{\phi} + \frac{3}{2}H\left(1 + 4\pi\eta\dot{\phi}^2\right),$$
$$S_5 = \frac{1}{2}\left(1 + 4\pi\eta\dot{\phi}^2\right),$$
$$S_6 = 4\pi\eta\dot{\phi}\ddot{\phi} + \frac{3}{2}H\left(1 + 4\pi\eta\dot{\phi}^2\right),$$
$$S_7 = \frac{1}{2}\left(1 + 4\pi\eta\dot{\phi}^2\right),$$
$$S_8 = -\frac{1 - 4\pi\eta\dot{\phi}^2}{2a^2}. \tag{31}$$

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Received 25.11.2014

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КОСМОЛОГИЧЕСКИЕ ВОЗМУЩЕНИЯ В ТЕОРИИ ГРАВИТАЦИИ С НЕМИНИМАЛЬНОЙ КИНЕТИЧЕСКОЙ СВЯЗЬЮ

Рассмотрены космологические возмущения в теории гравитации с неминимальной кинетической связью. Лагранжиан модели содержит член вида $\eta G^{ij}\phi_{,i}\phi_{,j}$ и представляет собой частный пример общего лагранжиана Хорндески, который приводит к уравнениям движения второго порядка. Построен полный набор уравнений для скалярных, векторных и тензорных возмущений. Детально исследованы тензорные моды возмущений. Показано, что их поведение под хаббловским горизонтом кардинально отличается от соответствующего поведения тензорных мод во фридмановской космологии.

Ключевые слова: *неминимальная кинетическая связь, космологические возмущения.*

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