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SUPERCONFORMAL STRUCTURES ON THE THREE-SPHERE

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This talk provides a brief account of the construction of the $2n$-extended supersphere $S^{3|4n}$, with $n = 1, 2, \ldots$, as a homogeneous space of the three-dimensional Euclidean superconformal group $\mathsf{OSp}(2n|2,2)$ such that its bosonic body is the three-sphere S^3.

Keywords: *superconformal group, $\mathcal{N}$-extended super three-sphere.*

1 Introduction

Recently, there has been an interest [1–3] in superconformal field theories on a three-dimensional (3D) sphere, mostly motivated by the study of their quantum features with the use of localization techniques. In addition to the issues raised in [1–3] and related papers, it is also of interest to study correlation functions in superconformal field theories on S^3, and a superspace setting appears to be most suitable to address this goal. An $\mathcal{N} = 2$ superspace formalism was developed to describe supersymmetric gauge theories on S^3 [4], but superconformal aspects of these and more general theories were not studied in the Euclidean superspace framework so far.

A geometric formalism required for constructing off-shell superconformal field theories on S^3 has been developed in a recent paper [5]. In that paper, we introduced a $2n$-extended supersphere $S^{3|4n}$, with $n = 1, 2, \ldots$, as a homogeneous space of the 3D Euclidean superconformal group, $\mathsf{OSp}(2n|2,2)$, with the property that the bosonic body of $S^{3|4n}$ is the three-sphere[1]. Supertwistor and bi-supertwistor realizations of $S^{3|4n}$ were derived. To some extent, these realizations are analogous to those of 3D and 4D compactified Minkowski superspaces $\overline{\mathbb{M}}^{3|2\mathcal{N}}$ and $\overline{\mathbb{M}}^{4|4\mathcal{N}}$, respectively, described in detail in [6–8]. However, the Euclidean case was shown [5] to have new nontrivial features. This talk provides a brief overview of the constructions given in [5].

2 The conformal and superconformal groups in three Euclidean dimensions

The conformal group of both the three-sphere S^3 and the Euclidean three-plane $\mathbb{E}^3$ is $\mathsf{SO}(4,1)$. The same group is also a subgroup of the isometry group $\mathsf{O}(4,1)$ of four-dimensional de Sitter space dS_4. Its connected component $\mathsf{SO}_0(4,1)$ is locally isomorphic[2] to the dS_4 spin group $\mathsf{USp}(2,2)$ defined by

$$\mathsf{USp}(2,2) = \mathsf{SU}(2,2) \bigcap \mathsf{Sp}(4,\mathbb{C}) \ . \tag{2.1}$$

Here $\mathsf{SU}(2,2)$ is a two to one covering group of the connected component $\mathsf{SO}_0(4,2)$ of the conformal group of four-dimensional Minkowski space $\mathbb{M}^4 = \mathbb{E}^{3,1}$,

$$\mathsf{SU}(2,2) := \Big\{ g \in \mathsf{SL}(4,\mathbb{C}) \ , \quad g^\dagger I g = I \ , \quad I = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix} \Big\} \ . \tag{2.2}$$

The symplectic group $\mathsf{Sp}(4,\mathbb{C})$ is realized as follows

$$\mathsf{Sp}(4,\mathbb{C}) := \Big\{ g \in \mathsf{GL}(4,\mathbb{C}) \ , \quad g^{\mathrm{T}} \Lambda g = \Lambda \ , \quad \Lambda = \begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix} \Big\} \ , \tag{2.3}$$

where σ_2 is the second Pauli matrix. The matrix Λ satisfies the properties

$$\Lambda^\dagger = -\Lambda^{\mathrm{T}} = \Lambda \ , \qquad \Lambda^2 = \mathbb{1}_4 \ . \tag{2.4}$$

The $\mathcal{N}$-extended superconformal group in three Euclidean dimensions is

$$\mathsf{OSp}(2n|2,2) = \mathsf{SU}(n,n|2,2) \bigcap \mathsf{OSp}(2n|4;\mathbb{C}) \ , \quad n = 1, 2, \ldots \ , \tag{2.5}$$

with $\mathcal{N} = 2n$. It consists of $(2n|4) \times (2n|4)$ supermatrices (with A, D bosonic blocks and B, C fermionic ones)

$$g = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) \tag{2.6}$$

[1] The supersphere $S^{3|4n}$ has $4n$ Grassmann-odd directions that are parametrized by $2n$ two-component spinor coordinates.

[2] The group $\mathsf{USp}(2,2)$ is a two to one covering group of $\mathsf{SO}_0(4,1)$.

constrained by

$$g^\dagger \Xi g = \Xi , \qquad \Xi = \left(\begin{array}{c|c} \Omega & 0 \\ \hline 0 & I \end{array}\right) , \qquad (2.7a)$$

$$\Omega^\dagger = -\Omega^{\rm T} = \Omega , \quad \Omega^2 = \mathbb{1}_{2n} ,$$

$$g^{\rm sT} \Upsilon g = \Upsilon , \qquad \Upsilon = \left(\begin{array}{c|c} \mathbb{1}_{2n} & 0 \\ \hline 0 & \Lambda \end{array}\right) , \qquad (2.7b)$$

$$g^{\rm sT} = \left(\begin{array}{c|c} A^{\rm T} & C^{\rm T} \\ \hline -B^{\rm T} & D^{\rm T} \end{array}\right) .$$

The bosonic subgroup of $\mathsf{OSp}(2n|2,2)$ is $\mathsf{SO}^*(2n) \times \mathsf{USp}(2,2)$. Here we define the group $\mathsf{SO}^*(2n)$ by

$$\mathsf{SO}^*(2n) := \Big\{ \mathfrak{U} \in \mathsf{GL}(2n,\mathbb{C}),\ \mathfrak{U}^{\rm T}\mathfrak{U} = \mathbb{1}_{2n},\ \mathfrak{U}^\dagger \Omega U = \Omega \Big\}.$$

This definition of $\mathsf{SO}^*(2n)$ is equivalent to a standard one. This group is the R-symmetry subgroup of the superconformal group $\mathsf{OSp}(2n|2,2)$. It is non-compact for $n > 1$. The case $n = 1$ is really special, since the R-symmetry group is compact, $\mathsf{SO}^*(2) \cong \mathsf{U}(1)$.

3 Twistor realization of the three-sphere

Introduce two $\mathsf{USp}(2,2)$ invariant inner products on $\mathbb{C}^4$:

$$\langle S|T\rangle_I := S^\dagger I\, T = \overline{S_{\hat\alpha}} I^{\hat\alpha\hat\beta} T_{\hat\beta} , \qquad (3.8a)$$

$$\langle S|T\rangle_\Lambda := S^{\rm T} \Lambda\, T = S_{\hat\alpha} \Lambda^{\hat\alpha\hat\beta} T_{\hat\beta} , \qquad (3.8b)$$

for any $T, S \in \mathbb{C}^4$. We will refer to this space as twistor space, and its elements will be called twistors. A twistor is viewed as a column vector

$$T = (T_{\hat\alpha}) = \left(\begin{array}{c} f_\alpha \\ g_\beta \end{array}\right) , \qquad (3.9)$$

with the two-component spinors f_α and g_β being complex.

Consider the space of all two-planes in $\mathbb{C}^4$ known as the Grassmannian $G_{2,4}(\mathbb{C})$. Any two-plane is determined by its basis, i.e. by two linearly independent twistors T^μ, with $\mu = 1, 2$. Such a basis $\{T^\mu\}$ is defined only modulo the equivalence relation

$$\{T^\mu\} \sim \{\tilde T^\mu\} , \ \tilde T^\mu = T^\nu R_\nu{}^\mu , \ R \in \mathsf{GL}(2,\mathbb{C}) . \quad (3.10)$$

Equivalently, the Grassmannian $G_{2,4}(\mathbb{C})$ can be thought of as consisting of all 4×2 complex matrices of rank two,

$$(T^1\ T^2) = \left(\begin{array}{c} F \\ G \end{array}\right) , \qquad (3.11)$$

where the 2×2 matrices F and G are defined modulo the equivalence relation

$$\left(\begin{array}{c} F \\ G \end{array}\right) \sim \left(\begin{array}{c} F\,R \\ G\,R \end{array}\right) , \qquad R \in \mathsf{GL}(2,\mathbb{C}) . \qquad (3.12)$$

Let $\mathfrak{S}$ denote the subspace of $G_{2,4}(\mathbb{C})$ consisting of all two-planes in $\mathbb{C}^4$ that are null with respects to the two inners products (3.8). For any two-plane belonging to $\mathfrak{S}$, it holds that

$$\langle T^\mu|T^\nu\rangle_I = 0 \ , \ \langle T^\mu|T^\nu\rangle_\Lambda = 0 \ , \ \mu,\nu = 1,2 \qquad (3.13)$$

or, equivalently,

$$F^\dagger F - G^\dagger G = 0 , \qquad (3.14a)$$

$$F^{\rm T}\sigma_2 F - G^{\rm T}\sigma_2 G = 0 . \qquad (3.14b)$$

It is known that the space of all two-planes in $\mathbb{C}^4$ under the null condition (3.14a) is compactified 4D Minkowski space, $\overline{\mathbb{M}}^4 = (S^3 \times S^1)/\mathbb{Z}_2$, see e.g. [7]. As shown in [7], the conditions that the 4×2 matrix (3.11) has rank two and obeys (3.14a) imply that

$$\det F \neq 0 \quad \text{and} \quad \det G \neq 0 . \qquad (3.15)$$

The equivalence relation (3.12) tells us that

$$\left(\begin{array}{c} F \\ G \end{array}\right) \sim \left(\begin{array}{c} h \\ \mathbb{1}_2 \end{array}\right) . \qquad (3.16)$$

Now the conditions (3.14a) and (3.14b) imply, respectively,

$$h^\dagger h = \mathbb{1}_2 \quad \Longrightarrow \quad h \in \mathsf{U}(2) ; \qquad (3.17a)$$

$$h^{\rm T}\sigma_2 h = \sigma_2 \quad \Longrightarrow \quad \det h = 1 . \qquad (3.17b)$$

We conclude that $\mathfrak{S}$ may be identified with the group manifold $\mathsf{SU}(2) = S^3$.

Given a group element

$$g = (g_{\hat\alpha}{}^{\hat\beta}) = \left(\begin{array}{cc} A & B \\ C & D \end{array}\right) \in \mathsf{USp}(2,2) , \qquad (3.18)$$

with A, B, C and D some 2×2 matrices, its action on S^3 is a fractional linear transformation

$$h \to h' = (Ah + B)(Ch + D)^{-1} . \qquad (3.19)$$

All null two-planes prove to be real with respect to a certain involution on the Grassmannian $G_{2,4}(\mathbb{C})$ [5].

4 The supersphere as a conformal superspace

The supergroup $\mathsf{OSp}(2n|2,2)$ naturally acts on the space of even supertwistors and also on the space of odd supertwistors. An arbitrary supertwistor looks like

$$T = (T_A) = \left(\begin{array}{c} T_i \\ T_{\hat\alpha} \end{array}\right) , \qquad i = 1, \dots, 2n . \qquad (4.20)$$

In the case of even supertwistors, T_i is fermionic and $T_{\hat\alpha}$ is bosonic. In the case of odd supertwistors, T_i is bosonic and $T_{\hat\alpha}$ is fermionic. We introduce the parity function $\varepsilon(T)$ defined as: $\varepsilon(T) = 0$ if T is even, and $\varepsilon(T) = 1$ if T is odd. We also define

$$\varepsilon_A = \begin{cases} 1 & A = i \\ 0 & A = \hat\alpha \end{cases} .$$

Then the above definition can be rewritten as

$$\varepsilon(T_A) = \varepsilon(T) + \varepsilon_A \pmod 2 \ . \tag{4.21}$$

Even and odd supertwistors are called pure. The space of even supertwistors may be identified with $\mathbb{C}^{4|2n}$.

Supertwistors transform in the defining representation of $\mathsf{OSp}(2n|2,2)$,

$$T \ \to \ T' = gT \ , \qquad g \in \mathsf{OSp}(2n|2,2) \ . \tag{4.22}$$

This transformation law implies that the supergroup $\mathsf{OSp}(2n|2,2)$ defined by (2.5)–(2.7) leaves invariant two inner products

$$\langle S|T\rangle_\Xi \ := \ S^\dagger\,\Xi\,T = \overline{S_A}\,\Xi^{AB}\,T_B \ , \tag{4.23a}$$

$$\langle S|T\rangle_\Upsilon \ := \ (-1)^{\varepsilon_A+\varepsilon(S)\cdot\varepsilon_A} S_A \Upsilon^{AB} T_B \ , \tag{4.23b}$$

for arbitrary pure supertwistors S and T. These inner products have the following fundamental properties:

$$\overline{\langle T_1|T_2\rangle_\Xi} \ = \ \langle T_2|T_1\rangle_\Xi \ ; \tag{4.24a}$$

$$\langle T_1|T_2\rangle_\Upsilon \ = \ -(-1)^{\varepsilon_1\varepsilon_2}\langle T_2|T_1\rangle_\Upsilon \ , \tag{4.24b}$$

for arbitrary pure supertwistors T_1 and T_2.

A dual supertwistor

$$Z = (Z^A) = \left(Z^i, Z^{\hat\alpha}\right) \ , \qquad i = 1, \dots, 2n \tag{4.25}$$

transforms under $\mathsf{OSp}(2n|2,2)$ such that $Z^A T_A$ is invariant for any supertwistor T,

$$Z \ \to \ Z' = Zg^{-1} \ , \qquad g \in \mathsf{OSp}(2n|2,2) \ . \tag{4.26}$$

A dual supertwistor Z is even (odd) if $Z^A T_A$ is a c-number for any even (odd) supertwistor T.

Invariance of the inner product (4.23b) under $\mathsf{OSp}(2n|2,2)$ tells us that

$$\begin{aligned} Z^A \ &:= \ -(1)^{\varepsilon_B+\varepsilon(S)\varepsilon_B} S_B \Upsilon^{BA} \\ &= \ (-1)^{\varepsilon(S)\varepsilon_A} \Upsilon^{AB} S_B \end{aligned} \tag{4.27}$$

is a pure dual supertwistor. Conversely, given a pure dual supertwistor Z^A, the following object

$$S_A := (-1)^{\varepsilon(Z)\varepsilon_B} (\Upsilon^{-1})_{AB} Z^B \tag{4.28}$$

is a pure supertwistor. We emphasize that Υ^{AB} is an invariant tensor of the superconformal group,

$$(g^{\rm sT})^A{}_C \Upsilon^{CD} g_D{}^B = \Upsilon^{AB} \ , \tag{4.29a}$$

$$(g^{\rm sT})^A{}_B = (-1)^{\varepsilon_A\varepsilon_B+\varepsilon_B} g_B{}^A \ , \tag{4.29b}$$

for any group element $g \in \mathsf{OSp}(2n|2,2)$.

Since the inner product (4.23a) is invariant under $\mathsf{OSp}(2n|2,2) \subset$, we observe that

$$\bar S^A := \overline{S_B}\,\Xi^{BA} \tag{4.30}$$

is a dual supertwistor, for any pure supertwistor S_A [3]. In conjunction with our previous result (4.28), this implies the existence of a one-to-one map of supertwistor space onto itself defined by

$$\star: \ S_A \to (\star S)_A := (-1)^{\varepsilon_C+\varepsilon(S)\varepsilon_C} (\Upsilon^{-1})_{AB} \Xi^{BC} \overline{S_C} \ , \tag{4.31}$$

for any pure supertwistor S_A. This map is characterized by the property

$$\star\star = -\mathbb{1}_{2n|4} \ , \tag{4.32}$$

which follows from the observations that the matrices Ω and ΛI (i) are purely imaginary; and (ii) fulfill the identities $\Omega^2 = \mathbb{1}_{2n}$ and $(\Lambda I)^2 = \mathbb{1}_4$.

We define a $2n$-extended supersphere $S^{3|4n}$ to be the space of all null and real two-planes in the space of even supertwistors $\mathbb{C}^{4|2n}$. In general, any two-plane in $\mathbb{C}^{4|2n}$ is generated by two supertwistors T^μ such that their bodies are linearly independent. Equivalently, it may be described by a rank-two $(2n|4)\times 2$ supermatrix

$$(T^\mu) = \begin{pmatrix} \Theta \\ \hline F \\ G \end{pmatrix} \ , \qquad \mu = 1, 2 \ , \tag{4.33}$$

which is defined modulo the equivalence relation

$$\begin{pmatrix} \Theta \\ \hline F \\ G \end{pmatrix} \sim \begin{pmatrix} \Theta\,R \\ \hline F\,R \\ G\,R \end{pmatrix} \ , \qquad R \in \mathsf{GL}(2,\mathbb{C}) \ . \tag{4.34}$$

Here Θ is a $2n \times 2$ fermionic matrix, and F and G are 2×2 bosonic matrices. The two-planes belonging to $S^{3|4n}$ are required to be (i) null with respect to the two inner products (4.23); and (ii) real with respect to the star-map (4.31) modulo the equivalence relation (4.34). The null conditions are

$$\Theta^\dagger\Omega\Theta + F^\dagger F - G^\dagger G \ = \ 0 \ ; \tag{4.35a}$$

$$-\Theta^{\rm T}\Theta + F^{\rm T}\sigma_2 F - G^{\rm T}\sigma_2 G \ = \ 0 \ . \tag{4.35b}$$

As in the bosonic case, the first null condition implies that $\det F \neq 0$ and $\det G \neq 0$. As a result, the null two-plane can equivalently be described by a supermatrix

$$\mathcal{P} = \begin{pmatrix} \Theta \\ \hline \boldsymbol{h} \\ \mathbb{1}_2 \end{pmatrix} = \begin{pmatrix} \Theta_i{}^\beta \\ \hline \boldsymbol{h}_\alpha{}^\beta \\ \delta_\gamma{}^\beta \end{pmatrix} \ , \tag{4.36}$$

where the null conditions (4.35) now read

$$\Theta^\dagger\Omega\Theta + \boldsymbol{h}^\dagger\boldsymbol{h} \ = \ \mathbb{1}_2 \ , \tag{4.37a}$$

$$-\Theta^{\rm T}\Theta + \boldsymbol{h}^{\rm T}\sigma_2\boldsymbol{h} \ = \ \sigma_2 \ . \tag{4.37b}$$

[3] Eq. (4.23a) can be rewritten in the form $\langle S|T\rangle_\Xi = \bar S^A T_A$.

The condition that the two-plane (4.36) is real under (4.31) amounts to

$$\overline{\Theta} = -\Omega\Theta\sigma_2 , \quad (4.38a)$$

$$\overline{\boldsymbol{h}} = \sigma_2\boldsymbol{h}\sigma_2 . \quad (4.38b)$$

Eq. (4.38a) is a pseudo-Majorana condition. The conditions (4.37) are super extensions of (3.17).

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СУПЕРКОНФОРМНАЯ СТРУКТУРА 3-СФЕРЫ

Настоящая работа представляет краткое описание построения $2n$-расширенной суперсферы $S^{3|4n}$, для $n = 1, 2, \ldots$, как однородного пространства трехмерной евклидовой суперконформной группы $\mathsf{OSp}(2n|2,2)$ такой, что ее бозонная часть является 3-сферой S^3.

Ключевые слова: *суперконформная группа, $\mathcal{N}$-расширенная супер 3-сфера.*

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