UDC 530.1; 539.1

THE CASIMIR ENERGY FOR TWO AND THREE LAYER OF GRAPHENS

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The Casimir energy of system of parallel conductive planes with constant conductivity is considered. General form of the Casimir energy for two and three planes is obtained. For the case of equal interplane distances the energy is proportional to inverse third power of distance. For small conductivity the energy does not depend on the Planck constant and velocity of light. The Casimir energy of planes with ideal conductivity is the sum of the Casimir energy of the neighboring planes.

Keywords: *Casimir energy, zeta-function, zero point energy, graphene.*

1 Introduction

Since the remarkable paper of Casimir [1] which was published more then 60 years ago, the significant progress was done both experimental justification and theoretical description of the effect which was called as Casimir effect. At the presence there are many reviews [2–5] and books [6, 7] in which this phenomenon is considered from different points of view. Interaction between bodies at short distances when retardation of electromagnetic field is neglected usually called as van der Waals interaction. At greater distances where retardation is important the interaction usually called the Casimir force. In particular case of two atom, the potential of interaction falls down as seventh power of distance for Casimir case and as sixth power for short distance.

In the paper we analyze in detail in the framework of zeta-function approach the Casimir energy for two and three parallel conductive planes. The real physical object for which these calculations have direct application is graphene – monolayer of carbon atoms [8]. Graphene is conductor with constant conductivity $\sigma = e^2/4$ (e – electron charge) up to energy $\mathcal{E}_t \sim 3эB$ (corresponding frequency $\omega_t \sim 4.5PHz$ and wavelength $\lambda_t \sim 413nm$). It was predicted theoretically [9] and experimentally established in Refs. [10, 11]. Here we consider simple model of conductivity – it is constant for any frequency.

The dimension of conductivity for two dimensional case is the same as velocity. For this reason in relativistic theory the dimensionless parameter $\eta = 2\pi\sigma/c$ appears which is for graphene $\pi e^2/2\hbar c = \pi\alpha/2 \approx 0.0115$, where α is the fine-structure constant.

Hereafter we use units with $\hbar = c = 1$.

2 TE and TM Electromagnetic modes

Let us set planes perpendicular to z axes. Due to symmetry of the problem we represent the field in the following form

$$\boldsymbol{E} = \boldsymbol{e}(z)e^{ik_x x+ik_y y-i\omega t},\ \boldsymbol{H} = \boldsymbol{h}(z)e^{ik_x x+ik_y y-i\omega t}.$$

The Maxwell equations out of the planes have no currencies and charges and we may express all components in terms on single component.

TE mode ($e_z = 0$):

$$\begin{aligned} e_x &= -\frac{\omega k_y}{ck_\perp^2}h_z,\ h_x = \frac{ik_x}{k_\perp^2}h_z', \\ e_y &= +\frac{\omega k_x}{ck_\perp^2}h_z,\ h_y = \frac{ik_y}{k_\perp^2}h_z', \\ h_z'' &= k^2 h_z. \end{aligned}$$

TM mode ($h_z = 0$):

$$\begin{aligned} e_x &= \frac{ik_x}{k_\perp^2}e_z',\ h_x = +\frac{\omega k_y}{ck_\perp^2}e_z, \\ e_y &= \frac{ik_y}{k_\perp^2}e_z',\ h_y = -\frac{\omega k_x}{ck_\perp^2}e_z, \\ e_z'' &= k^2 e_z. \end{aligned}$$

Here $k_\perp^2 = k_x^2 + k_y^2$, $k^2 = k_\perp^2 - \omega^2$ and prime means derivative with respect of z.

The boundary conditions at the plane at $z = a$ read

$$\begin{aligned} \mathrm{TE} &: [h_z]_r = 0,\ [h_z']_r = 4\pi i\sigma\omega h_z|_r, \\ \mathrm{TM} &: [e_z']_r = 0,\ [e_z]_r = -\frac{4\pi i\sigma}{\omega}e_z'|_r, \end{aligned} \qquad (1)$$

where $[f(z)]_r = f(r-0) - f(r+0)$.

Therefore for each plane we have two equations for each modes. For n parallel planes we have $2n$ equations. There are $n-1$ intervals between planes and in each gap there are two constants in general solution. Therefore, summary amount of constants is $2n-2$. There are also two half-subspaces out of planes and in each subspace solution is defined by single constant. Totally we $2n$ constants obeying $2n$ equations. Solution of this system exists if and only if the main determinant equals to zero. This condition allow us to obtain in manifest form the function Ψ which we need to calculate contribution of each mode to the energy (see [7])

$$\mathcal{E}_{\mathsf{TE},\mathsf{TM}}(s) = -M^{2s}\frac{\cos\pi s}{2\pi}\iint\frac{d^2k}{(2\pi)^2}$$
$$\times\int_0^\infty d\lambda\lambda^{1-2s}\frac{\partial}{\partial\lambda}\ln\Psi_{\mathsf{TE},\mathsf{TM}}(i\lambda). \tag{2}$$

The total energy is sum $\mathcal{E}(s) = \mathcal{E}_{\mathsf{TE}}(s) + \mathcal{E}_{\mathsf{TM}}(s)$.

3 Two planes

For two planes we obtain

$$\mathcal{E}^{(2)} = \frac{Q^{(2)}(\eta)}{d^3},\ Q^{(2)} = Q^{(2)}_{\mathsf{TE}} + Q^{(2)}_{\mathsf{TM}}, \tag{3}$$

where

$$Q^{(2)}_{\mathsf{TE}}(\eta) = \frac{1}{32\pi^2}\int_0^\infty y^2dy\int_0^1 dx\ln\left(1-\frac{\eta^2x^2e^{-y}}{(1+\eta x)^2}\right),$$
$$Q^{(2)}_{\mathsf{TM}}(\eta) = \frac{1}{32\pi^2}\int_0^\infty y^2dy\int_0^1 dx\ln\left(1-\frac{\eta^2e^{-y}}{(x+\eta)^2}\right).$$

Therefore, the Casimir energy proportional to inverse third power of distance between planes.

In the limit of infinite conductivity (ideal conductor) $\eta\to\infty$ we obtain

$$\mathcal{E}^{(2)} = -\frac{\pi^2}{720d^3}, \tag{4}$$

in full agreement with Casimir result. For small conductivity, $\eta = 2\pi\sigma/c \to 0$ the main contribution comes from TM mode:

$$\mathcal{E}^{(2)} = -\frac{a_2\eta}{2\pi d^3} = -\frac{a_2\sigma}{d^3}, \tag{5}$$

where

$$a_2 = -\frac{1}{16\pi}\int_0^\infty y^2dy\int_0^\infty dx\ln\left(1-\frac{e^{-y}}{(1+x)^2}\right)$$
$$\approx 0.0407509\ldots,$$

and $\zeta_R(a)$ is Riemann zeta-function. TE mode gives quadratic over conductivity contribution. For graphene conductivity we obtain

$$\mathcal{E}^{(2)} = -\frac{e^2b}{32\pi d^3}, \tag{6}$$

where $b = 8\pi a_2 \approx 1.0241\ldots$. It is important to note that this expression has the same form in dimensional variables and it does not depend on the Planck constant and velocity of light. This resul was obtained in Refs. [12,13] on the basis of different calculations.

4 Three planes

For three plane with equal interplane distance $d_{21} = d_{32} = d$ we obtain the following expression for energy

$$\mathcal{E}^{(3)} = \frac{Q^{(3)}(\eta)}{d^3},\ Q^{(3)} = Q^{(3)}_{\mathsf{TE}} + Q^{(3)}_{\mathsf{TM}}, \tag{7}$$

where

$$Q^{(3)}_{\mathsf{TE}} = \frac{1}{32\pi^2}\int_0^\infty y^2dy\int_0^1 dx$$
$$\times\ln\left(1-\frac{\eta^2x^2e^{-y}}{(1+\eta x)^2}\left\{2+\frac{1-\eta x}{1+\eta x}e^{-y}\right\}\right),$$
$$Q^{(3)}_{\mathsf{TM}} = \frac{1}{32\pi^2}\int_0^\infty y^2dy\int_0^1 dx$$
$$\times\ln\left(1-\frac{\eta^2e^{-y}}{(x+\eta)^2}\left\{2+\frac{x-\eta}{x+\eta}e^{-y}\right\}\right).$$

For small conductivity $\eta\ll 1$ one has

$$\mathcal{E}^{(3)} = -\frac{a_3\eta}{2\pi d^3} = -\frac{a_3\sigma}{d^3}, \tag{8}$$

where

$$a_3 = -\frac{1}{16\pi}\int_0^\infty y^2dy\int_0^\infty dx$$
$$\times\ln\left(1-\frac{e^{-y}}{(x+1)^2}\left\{2+\frac{x-1}{x+1}e^{-y}\right\}\right) = 0.0832892\ldots.$$

Let us consider influence third plane for energy of two planes. We normalize distances on the distance between first and second planes $d = d_{21}$ and extract the part corresponding for two planes (3):

$$\mathcal{E}^{(3)} = \mathcal{E}^{(2)}(d)\triangle(\eta, q_{32}),$$
$$\triangle(\eta, q_{32}) = \frac{Q^{(3)}_{\mathsf{TE}}(\eta, q_{32}) + Q^{(3)}_{\mathsf{TM}}(\eta, q_{32})}{Q^{(2)}_{\mathsf{TE}}(\eta) + Q^{(2)}_{\mathsf{TM}}(\eta)},$$

where

$$Q^{(3)}_{\mathsf{TE}} = \frac{1}{32\pi^2}\int_0^\infty y^2dy\int_0^1 dx\ln\left(1-\frac{\eta^2x^2}{(1+\eta x)^2}\right.$$
$$\left.\left\{e^{-y}+e^{-q_{32}y}+\frac{1-\eta x}{1+\eta x}e^{-(1+q_{32})y}\right\}\right),$$
$$Q^{(3)}_{\mathsf{TM}} = \frac{1}{32\pi^2}\int_0^\infty y^2dy\int_0^1 dx\ln\left(1-\frac{\eta^2}{(x+\eta)^2}\right.$$
$$\left.\left\{e^{-y}+e^{-q_{32}y}+\frac{x-\eta}{x+\eta}e^{-(1+q_{32})y}\right\}\right),$$

and $q_{32} = d_{32}/d$.

Numerical calculation shows that $\triangle$ depends weak on the conductivity if third plane is in the distance greater then distance between first and second planes $d_3 - d_2 \geq d$.

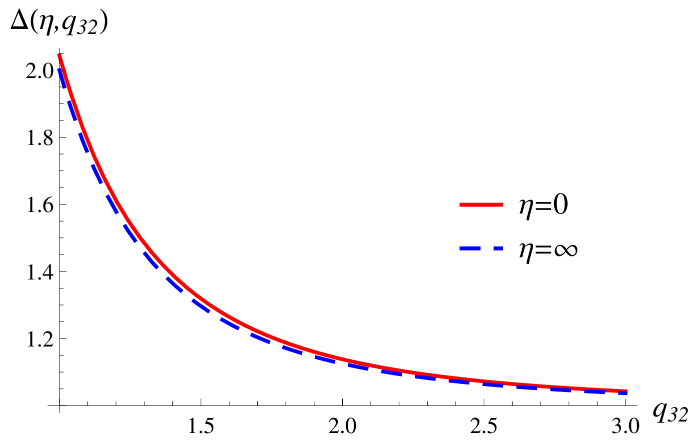

Figure 1. Dependence $\triangle$ on the dimensionless distance $q_{32} = d_{32}/d$ (d is distance between 1st and 2nd planes, d_{32} – distance between 2nd and 3rd planes) for two limit cases of conductivity $\eta = 0$ and $\eta = \infty$ (ideal case)

If third plane is closer the situation is changed and main contribution comes from second and third planes and first plane plays the role of additional third plane. Dependence of the $\triangle$ is shown on the Fig. 1 for two limit cases of conductivity $\eta = 0$ and $\eta = \infty$ (ideal case). It is easy to see that curves weakly differ each other.

Let us set third plane far from first two planes $q_{32} \gg 1$. In this case

$$\triangle \approx 1 + \frac{A(\eta)}{q_{32}^3},$$

and

$$\mathcal{E}^{(3)} = \mathcal{E}^{(2)}(d_{21}) + A(\eta)\mathcal{E}^{(2)}(d_{32}).$$

Here

$$A(\eta) = \int_0^\infty y^2 dy \int_0^\infty dx \ln\left\{\left(1 - \frac{2\eta^2 x^2 e^{-y}}{(1+\eta x)(1+2\eta x)}\right) \times \left(1 - \frac{2\eta^2 e^{-y}}{(x+\eta)(x+2\eta)}\right)\right\} / \int_0^\infty y^2 dy \int_0^\infty dx \times \ln\left\{\left(1 - \frac{\eta^2 x^2 e^{-y}}{(1+\eta x)^2}\right)\left(1 - \frac{\eta^2 e^{-y}}{(x+\eta)^2}\right)\right\}.$$

The function $A(\eta)$ monotone tends to unit $A(\eta)_{\eta\to\infty} = 1$ starting with $A(0) = 1.3857\ldots$. Therefore, third plane gives additive contribution to the energy with weight $A(\eta)$.

5 Conclusion

Let us summarize the results obtained. We considered parallel conductive planes with constant conductivity. We made calculations in manifest form for two and three planes. Because conductivity σ has dimension of light it is combined with velocity of light to dimensionless combination $\eta = 2\pi\sigma/c$. For the case of same distances between planes there is the only parameter with dimension of length – distant between planes. Therefore, the energy for n parallel planes should get the following form

$$\mathcal{E}^{(n)} = \frac{\hbar c}{d^3} Q_n(\eta).$$

In the case of small conductivity $\eta \ll \sigma$ we obtain

$$\mathcal{E}^{(n)} \approx -\hbar \frac{a_n \sigma}{d^3},$$

where $a_2 = 0.0407\ldots$, $a_3 = 0.0832\ldots$. For the case of graphene sheets the conductivity $\sigma = e^2/4\hbar$ the energy does not depend on the Planck constant and velocity of light

$$\mathcal{E}^{(n)} \approx -\frac{a_n e^2}{4d^3}.$$

In the ideal case $\sigma \to \infty$ the Casimir energy is the sum of Casimir energy of neighboring planes. The finite conductivity breaks this simple dependence and the energy becomes complex function of distances d_{ik} between planes i and k.

Acknowledgement

This research has been supported in the part by the grant for RFBR, project No. 13-02-00757-a.

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Received 12.11.2014

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ЭНЕРГИЯ КАЗИМИРА ДЛЯ ДВУХ И ТРЕХ СЛОЕВ ГРАФЕНА

Рассмотрена энергия Казимира для системы двух и трех параллельных плоскостей с постоянной проводимостью. Получено общее выражение для энергии Казимира для двух и трех плоскостей. В случае одинакового расстояния между плоскостями энергия обратно пропорциональна третьей степени расстояния между плоскостями. Для малой проводимости энергия не зависит от постоянной Планка и скорости света. Энергия Казимира для плоскостей с идеальной проводимостью равна сумме энергий Казимира соседних плоскостей.

Ключевые слова: *энергия Казимира, дзета-функция, энергия нулевых колебаний, графен.*

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