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TOWARDS REALISTIC MODELS FROM HIGHER-DIMENSIONAL THEORIES WITH FUZZY EXTRA DIMENSIONS

D. Gavriil[1], G. Manolakos[1], G. Zoupanos[1,2]

[1] Physics Department, National Technical University, Zografou Campus, GR-15780 Athens, Greece
[2] Theory Group, Physics Department Cern, Geneva, Switzerland

E-mail: d.gavriil@yahoo.gr , giorgismanolakos@gmail.com , George.Zoupanos@cern.ch

We briefly review the Coset Space Dimensional Reduction (CSDR) programme and the best model constructed so far and then we present some details of the corresponding programme in the case that the extra dimensions are considered to be fuzzy. In particular, we present a four-dimensional $\mathcal{N}=4$ Super Yang Mills theory, orbifolded by $\mathbb{Z}_3$, which mimics the behaviour of a dimensionally reduced $\mathcal{N}=1$, 10-dimensional gauge theory over a set of fuzzy spheres at intermediate high scales and leads to the trinification GUT $SU(3)^3$ at somehow lower, which in turn can be spontaneously broken to the MSSM in low scales.

Keywords: *coset space dimensional reduction, unification, fuzzy spheres, orbifold projection.*

1 Introduction

Since 1970's there has been an intense pursuit of unification, that is the establishment of a single theoretical model describing all interactions. Profound research activity has resulted in two very interesting frameworks, namely Superstring Theories [1] and Non-Commutative Geometry [2]. Both approaches, although developing independently, share common unification targets and aim at exhibiting improved renormalization properties in the ultraviolet regime as compared to ordinary field theories. Moreover, these two (initially) different frameworks were bridged together after realizing that a Non-Commutative gauge theory can describe the effective physics on D-branes whilst a non-vanishing background antisymmetric field is present [3].

Significant progress has recently been made regarding the dimensional reduction of the $E_8 \times E_8$ Heterotic String using non-symmetric coset spaces [4]-[20], in the presence of background fluxes and gaugino condensates. It is widely known that the large number of Standard Model's free parameters which enter the theory, because of the ad hoc introduction of the Higgs and Yukawa sectors, is a major problem demanding solution. This embarrassment can be overcome by considering that those sectors originate from a higher dimensional theory. Various frameworks starting with the Coset Space Dimensional Reduction (CSDR) [21–23] and the Scherk-Schwarz [24] reduction schemes suggest that unification of the gauge and Higgs sectors can take place making use of higher dimensions. This means that the four-dimensional gauge and Higgs fields are the surviving components of the reduction procedure of the gauge fields of a pure higher-dimensional gauge theory. Furthermore, the addition of fermions in the higher-dimensional gauge theory leads naturally (after CSDR) to Yukawa couplings in four dimensions. The last step in this unified description in high dimensions is to relate the gauge and fermion fields, which can be achieved by demanding that the higher-dimensional gauge theory is $\mathcal{N}=1$ supersymmetric, i.e. the gauge and fermion fields are members of the same vector supermultiplet. In order to maintain an $\mathcal{N}=1$ supersymmetry after dimensional reduction, Calabi-Yau (CY) manifolds serve as suitable compact internal spaces [25]. However, the moduli stabilization problem that arose, led to the study of compactification with fluxes (for reviews see e.g. [26]). Within the context of flux compactification, the recent developments suggested the use of a wider class of internal spaces, called manifolds with SU(3)-structure. The latter class of manifolds admits a nowhere-vanishing, globally-defined spinor, which is covariantly constant with respect to a connection with torsion and not with respect to the Levi-Civita connection as in the CY case. Here we focus on an interesting class of SU(3)-structure manifolds called nearly-Kähler manifolds.

The homogeneous nearly-Kähler manifolds in six dimensions have been classified in [27] and they are the three non-symmetric coset spaces $G_2/SU(3)$, $Sp(4)/(SU(2)\times U(1))_{non-max}$ and $SU(3)/U(1)\times U(1)$ and the group manifold $SU(2)\times SU(2)$. The latter cannot lead to chiral fermions in four dimensions and therefore, for our purposes, it is ruled out of further interest. It is worth noting that four-dimensional theories resulting from the dimensional reduction of ten-dimensional $\mathcal{N}=1$ supersymmetric gauge theories over non-symmetric coset spaces, contain terms which could be interpreted as soft scalar masses. Here we will briefly describe the dimensional reduction of the $\mathcal{N}=1$ supersymmetric E_8 gauge theory over the nearly-

Kähler manifold $SU(3)/U(1)\times U(1)$. More specifically, an extension of the Minimal Supersymmetric Standard Model (MSSM) was derived by dimensionally reducing the $E_8 \times E_8$ gauge sector of the heterotic string [28].

Non-Commutative geometry is considered as an appropriate framework for regularizing quantum field theories, or even better, building finite ones. Unfortunately, constructing quantum field theories on Non-Commutative spaces is much more difficult than expected and, furthermore, they present problematic ultraviolet features [29], see however [30] and [31]. In the beginning, several models of the type of Standard Model were built making use of the Seiberg-Witten map, but they could only be considered as effective theories which were also lacking renormalizability. A more promising use of Non-Commutative geometry in particle physics occurred after the suggestion that it would describe the extra dimensions [32]; see also [33]. This proposal motivated the construction of higher-dimensional models which present many interesting features, i.e. renormalizability, potential predictivity, etc.

In this framework has been developed a higher-dimensional gauge theory in which those extra dimensions are described by fuzzy spaces [32], i.e. matrix approximations on smooth manifolds. The first step was to find a manifold on which one would construct a higher-dimensional gauge theory. The appropriate one was the product of Minkowski space and a fuzzy coset space $(S/R)_F$. Afterwards, in order to achieve the necessary dimensional reduction, was made use of the CSDR scheme, which is described in the previous section. Although the reduction is performed using the CSDR programme, there is a significant difference between the ordinary and the fuzzy version: the four-dimensional gauge group that appears in the first, between the geometrical and the spontaneous breaking due to the four-dimensional Higgs fields, does not appear in the latter. In the fuzzy CSDR scheme, the spontaneous symmetry breaking occurs after solving the fuzzy CSDR constraints, resulting in a non-zero minimum of the four-dimensional potential. Thus, in four dimensions, there remains only one scalar field, the physical Higgs field, which does survive the spontaneous symmetry breaking. In the same way, regarding the Yukawa sector, we have the welcoming results of massive fermions as well as interactions among the physical Higgs field and fermions (Yukawa interactions). We conclude that in order to be able to reproduce the spontaneous symmetry breaking of the SM in this framework, one would have to consider large extra dimensions. A determinant difference between ordinary and fuzzy CSDR is that a non-Abelian gauge group G is not necessary in the higher-dimensions theory. The non-Abelian gauge theories in four dimensions can originate from a $U(1)$ group in the higher-dimensional theory.

These theories are equipped with a very strong advantage when compared to the rest higher-dimensional ones, that is renormalizability. Arguments leading to this result are given in [32], but the strongest one was given after examining the issue from a different perspective. In a detailed analysis, it was established a renormalizable four-dimensional $SU(N)$ gauge theory in which we assigned a scalar multiplet which dynamically develops fuzzy extra dimensions, forming a fuzzy sphere [34]. The model develops non-trivial vacua which are interpreted as 6-dimensional gauge theory, in which geometry and gauge group depend on the parameters that are present in the initial Lagrangian. We result with a finite tower of massive Kaluza-Klein modes, a result consistent with a dimensionally reduced higher-dimensional gauge theory. This model presents many interesting features. First, the extra dimensions are generated dynamically by a geometrical mechanism. This feature is based on a result from non-commutative gauge theory, namely that solutions of matrix models can be interpreted as non-commutative, or fuzzy, spaces. The above mechanism is very generic and does not need fine-tuning, which means that supersymmetry is not involved. In the renormalizable quantum field theory framework, this constitutes a realization of the concepts of compactification and dimensional reduction. Moreover, since it is a large N gauge theory, every analytical technique of this context should be available to be applied. More specifically, it proves that the general gauge group in low energies is $SU(n_1) \times SU(n_2) \times U(1)$ or $SU(n)$. In this model, gauge groups that are formed by more than two simple groups (apart from $U(1)$) are not observed.

The features that emerged from the above mechanism are quite appealing, suggesting the construction of phenomenologically viable models in particle physics. When addressed to this direction, one encounters a severe problem, that is the chiral-fermion assignment in four dimensions. The best candidates of the above category of models, when it comes to inserting the fermions, are theories with mirror fermions in bi-fundamental representations of the low energy gauge group [35]. Detailed studies on fermionic sectors of models, which obey the mechanism of dynamical generation of the extra dimensions with the fuzzy sphere or a product of two fuzzy spheres, showed that when extrapolating to low-energy, the fermionic sector of the theory consists of two mirror sectors, even after the inclusion of the magnetic fluxes on the two fuzzy spheres [36]. Although the presence of mirror fermions does not exclude the possibility to obtain phenomenologically viable models [37], it is certainly preferred to end up with exactly chiral fermions. This is achieved by extending the above context and by inserting an additional structure which is based on orbifolds. Specifically, a $\mathbb{Z}_3$ orbifold projection of a $\mathcal{N} = 4$ $SU(3N)$ SYM theory leads to a $\mathcal{N} = 1$ supersymmetric theory with the gauge group being the $SU(3)^3$ [38]. In order to obtain specific vacua in the $\mathcal{N} = 1$ theory, required by interpreting the theory as resulting from fuzzy extra dimensions, one is normally obliged to introduce soft breaking supersymmetric terms. This induces the dynamical generation of twisted fuzzy spheres. The introduction of such soft breaking terms seems necessary in order to build phenomenologically viable supersymmetric theories, with MSSM being a very leading case. The vacua that emerge give rise to models which preserve the features described above, but also they accommodate a chiral low-energy spectrum. The most appealing chiral models of this kind are $SU(4)\times SU(2)\times SU(2)$, $SU(4)^3$ and $SU(3)^3$. The most interesting of those unified theories seems to be the latter, which is described by the trinification group. In addition, this theory can be upgraded to a two-loop finite theory (for reviews see [39–42]) and moreover it is able to make testable predictions [42]. Therefore, we conclude that fuzzy extra dimensions can be used in constructing chiral, renormalizable and phenomenologically viable field-theoretical models.

2 The coset space dimensional reduction

In the Coset Space Dimensional Reduction (CSDR) scheme (see [21–23] for a detailed exposition) one starts with a Yang-Mills-Dirac Lagrangian, with gauge group G, defined on a D-dimensional spacetime M^D, which is compactified to $M_4 \times S/R$ with S/R a coset space. S acts as a symmetry group on the extra coordinates and both S and its subgroup R, are Lie groups. As far as the most general S-invariant metric is concerned, it is always diagonal and depends on the number of radii that each space admits. Regarding the coset of our interest $(SU(3)/U(1)\times U(1))$ three radii R_1, R_2, R_3 are introduced. According to the CSDR framework, an S-transformation of the extra d coordinates is a gauge transformation of the fields that are defined on $M_4 \times S/R$, thus a gauge invariant Lagrangian written on this space is independent of the extra coordinates. Fields defined in this way are called symmetric. The initial gauge field $A_M(x,y)$ is split into its components $A_\mu(x,y)$ and $A_a(x,y)$, corresponding to M^4 and S/R respectively. Consider the action of a D-dimensional Yang-Mills theory with gauge group G, coupled to fermions defined on a manifold M^D compactified on $M_4 \times S/R$, $D = 4 + d, d = dimS - dimR$:

$$A = \int d^4x d^d y \sqrt{-g} \times \left[-\frac{1}{4} Tr(F_{MN}F_{K\Lambda}) g^{MK} g^{N\Lambda} + \frac{i}{2} \bar{\Psi} \Gamma^M D_M \Psi \right], \quad (1)$$

where $D_M = \partial_M - \theta_M - A_M$, with $\theta_M = \frac{1}{2}\theta_{MN\Lambda}\Sigma^{MN}$ the spin-connection of M^D, $F_{MN} = \partial_M A_N - \partial_N A_M - [A_M, A_N]$, where M, N run over the D-dimensional space and A_M and Ψ are D-dimensional symmetric fields. Let $\xi^\alpha_A, (A = 1, ..., dimS$ and $\alpha = dimR + 1, ..., dimS$ the curved index) be the Killing vectors which generate the symmetries of S/R and W_A the compensating gauge transformation associated with ξ_A. The requirement that transformations of the fields under the action of S/R are compensated by gauge transformations, is expressed by the following constraint equations for scalar ϕ, vector A_α and spinor ψ fields on S/R,

$$\delta_A \phi = \xi^\alpha_A \partial_\alpha \phi = D(W_A)\phi, \quad (2)$$

$$\delta_A A_\alpha = \xi^\beta_A \partial_\beta A_\alpha + \partial_\alpha \xi^\beta_A A_\beta = \partial_\alpha W_A - [W_A, A_\alpha], \quad (3)$$

$$\delta_A \psi = \xi^\alpha_A \partial_\alpha \psi - \frac{1}{2} G_{Abc} \Sigma^{bc} \psi = D(W_A)\psi\,, \quad (4)$$

where W_A depend only on internal coordinates y and $D(W_A)$ represents a gauge transformation in the appropriate representation of the fields.

The constraints (2)–(4) provide us [21, 22] with the four-dimensional unconstrained fields as well as with the gauge invariance that remains in the theory after dimensional reduction. The analysis of these constraints implies that the components $A_\mu(x,y)$ of the initial gauge field $A_M(x,y)$ become, after dimensional reduction, the four dimensional gauge fields and furthermore they are independent of y. In addition, one can find that they have to commute with the elements of the R_G, subgroup of G. Thus, the four-dimensional gauge group H is the centralizer of R in G, $H = C_G(R_G)$. The $A_\alpha(x,y)$ components of $A_M(x,y)$ denoted by $\phi_\alpha(x,y)$ from now on, become scalars in four dimensions and they transform under R as a vector υ, i.e.

$$S \supset R \quad (5)$$

$$adjS = adjR + \upsilon. \quad (6)$$

Furthermore, $\phi_\alpha(x,y)$ act as an interwining operator connecting induced representations of R acting on G and S/R. This implies, according to Schur's lemma, that the transformation properties of the fields $\phi_\alpha(x,y)$ under H can be found, if we express the adjoint representation of G in terms of $R_G \times H$:

$$G \supset R_G \times H \quad (7)$$

$$adjG = (adjR, 1) + (1, adjH) + \sum (r_i, h_i). \quad (8)$$

Then, if $\upsilon = \sum s_i$, where each s_i is an irreducible representation of R, there survives a Higgs multiplet transforming under the representation h_i of H. All other scalar fields vanish.

The analysis of the constraints imposed on spinors [22, 43–45] is analogous to the scalar cases and implies that the spinor fields act as interwining operators connecting induced representations of R in $SO(d)$ and in G. In order to specify the representation of H under which the four-dimensional fermions transform, we have to decompose the representation F of the initial gauge group in which the fermions are assigned in higher dimensions under $R_G \times H$, i.e.

$$F = \sum (t_i, h_i) \tag{9}$$

and the spinor of $SO(d)$ under R

$$\sigma_d = \sum \sigma_j. \tag{10}$$

It turns out that for each pair (r_i, σ_i), where r_i and σ_i are identical irreducible representations of R, there is an h_i multiplet of spinor fields in the four-dimensional theory. Regarding the existence of chiral fermions in the effective theory, we notice that if we start with Dirac fermions in higher dimensions it is impossible to obtain chiral fermions in four dimensions. Further requirements must be imposed in order to achieve chiral fermions in the resulting theory. Imposing the Weyl condition in D dimensions, we obtain two sets of Weyl fermions with the same quantum numbers under H. This is already a chiral theory, but still one can go further and try to impose Majorana condition in order to eliminate the doubling of the fermionic spectrum. Majorana and Weyl conditions are compatible in $D = 4n + 2$, which is the case of our interest.

An important requirement is that the resulting four-dimensional theories should be anomaly free. Starting with an anomaly free theory in higher dimensions, Witten [46] has given the condition to be fulfilled in order to obtain anomaly free four-dimensional theories. The condition restricts the allowed embeddings of R into G by relating them with the embedding of R into $SO(6)$, the tangent space of the six-dimensional cosets we consider [22, 47]. According to ref. [47] the anomaly cancellation condition is automatically satisfied for the choice of embedding

$$E_8 \supset SO(6) \supset R, \tag{11}$$

which we adopt here.

2.1 Dimensional Reduction of E_8 over $SU(3)/U(1) \times U(1)$

Let us next present a few results concerning the dimensional reduction of the $\mathcal{N} = 1, E_8$ SYM over $SU(3)/U(1) \times U(1)$ [48]. To determine the four-dimensional gauge group, the embedding of $R = U(1) \times U(1)$ in E_8 is suggested by the decomposition

$$E_8 \supset E_6 \times SU(3) \supset E_6 \times U(1)_A \times U(1)_B. \tag{12}$$

After the dimensional reduction of E_8 under $SU(3)/U(1) \times U(1)$, according to the rules of the previous section, the surviving gauge group in four dimensions is

$$\begin{aligned} H &= C_{E_8}(U(1)_A \times U(1)_B) \\ &= E_6 \times U(1)_A \times U(1)_B. \end{aligned} \tag{13}$$

Similarly, the explicit decomposition of the adjoint representation of E_8, 248 under $U(1)_A \times U(1)_B$ provides us with the surviving scalars and fermions in four dimensions. Eventually, one finds that the dimensionally reduced theory in four dimensions is a $\mathcal{N} = 1$, E_6 GUT with $U(1)_A, U(1)_B$ as global symmetries. The potential is determined by a tedious calculation [49, 50]. The D-terms can be constructed and the F-terms are obtained by the superpotential. The rest of the terms in the potential could be interpreted as soft scalar masses and trilinear soft terms. Finally, the gaugino mass was also calculated and receives contribution from the torsion contrary to the rest soft supersymmetry breaking terms.

2.2 $SU(3)^3$ due to Wilson flux

In order to reduce further the gauge symmetry, one has to apply the Wilson flux breaking mechanism [51–53]. Instead of considering a gauge theory on $M^4 \times B_0$ (B_0 a simply connected manifold in our case), one considers a gauge theory on $M^4 \times B$, with $B = B_0/F^{S/R}$ and $F^{S/R}$ a freely acting discrete symmetry of B_0. The discrete symmetries $F^{S/R}$, which act freely on coset spaces $B_0 = S/R$, are the center of S, $Z(S)$ and $W = W_S/W_R$, where W_S and W_R are the Weyl groups of S and R, respectively. In the case of our interest

$$F^{S/R} = \mathbb{Z}_3 \subseteq W. \tag{14}$$

The presence of the Wilson lines imposes further constraints on the fields of the theory. The surviving fields are invariant under the combined action of the discrete group $\mathbb{Z}_3$ on the geometry and on the gauge indices.

After the $\mathbb{Z}_3$ projection, the gauge group E_6 breaks to

$$SU(3)_C \times SU(3)_L \times SU(3)_R \tag{15}$$

(the first of the $SU(3)$ factors is the Standard Model colour gauge group). Moreover, one can obtain three fermion generations by introducing non-trivial monopole charges in the $U(1)$'s in R.

In ref [28] it was shown that the scalar potential leads to the proper hierarchy of spontaneous breaking. Using the appropriate vev's, a first spontaneous symmetry breaking leads to the MSSM [54], while the electroweak breaking proceeds by a second one [42]. It is worth noting that before the EW symmetry breaking, supersymmetry is broken by both D-terms and F-terms, in addition to its breaking by the soft terms.

We plan to examine in detail the phenomenological consequences of the resulting model, taking also into account the massive Kaluza Klein modes.

3 Field theory orbifolds and fuzzy spheres

Let us begin with reminding briefly how the orbifold structure applies in field theory, and how this structure is related to the dynamical generation of fuzzy extra dimensions. The reason is that we seek to end up with chiral fermions in the case of construction of models in particle physics.

We commence with a $SU(3N)$ $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory. The orbifold projection of this theory will be achieved by the action of the (discrete) group $\mathbb{Z}_3$. The procedure is to embed a discrete symmetry into the R-symmetry of the original theory, i.e. $SU(4)_R$. Due to this embedding, the projected theories we may end up with, may have different amount of remnant supersymmetry [55]. For example, supersymmetry is completely broken when $\mathbb{Z}_3$ is embedded maximally in $SU(4)_R$, while if it is embedded in an $SU(3)$ or $SU(2)$ subgroup of the $SU(4)_R$, it results to $\mathcal{N} = 1$ or $\mathcal{N} = 2$ supersymmetric theories, respectively. In this contribution, we concentrate on the $\mathcal{N} = 1$ case, which is compatible with our prime motivation, that is the construction of chiral models.

Projecting the initial theory under the discrete symmetry $\mathbb{Z}_3$ leads to a $\mathcal{N} = 1$ SYM theory, in which the only fields that remain are the ones that are invariant under the action of the discrete group, $\mathbb{Z}_3$. For the technicalities of this procedure see [38]. In the initial $\mathcal{N} = 4$ SYM theory, there are totally four superfields, one vector and three chiral in $\mathcal{N} = 1$ language. The component fields are the gauge fields $A_\mu, \mu = 0, \ldots 3$ of the $SU(3N)$ gauge group, three complex scalar fields $\phi^i, i = 1, \ldots 3$, which are accommodated in the adjoint of the gauge group and in the vector of the global symmetry and four Majorana fermions ψ^p, which are assigned in the adjoint of the gauge group and the spinor of the global symmetry. After the orbifold pojection, we end up with a theory which has different gauge group and particle spectrum. In short, $\mathbb{Z}_3$ acts non-trivially on the various fields depending on their representations under the R-symmetry and the gauge group [55]. The gauge group breaks down to $H = S(U(N) \times U(N) \times U(N))$ and the scalar and fermionic fields that survive, transform under the representations

$$3\left((N\bar{N}, 1) + (\bar{N}, 1, N) + (1, N, \bar{N})\right) \tag{16}$$

of the non-Abelian factor gauge groups, obtaining a spectrum free of gauge anomalies. It is easily understood that fermions belong to chiral representations and that there is a threefold replication, meaning there are three chiral families.

As for the F-term scalar potential of the $N = 4$ SYM theory, we obtain

$$V_F(\phi) = \frac{1}{4}\mathrm{Tr}\left(\left[\phi^i, \phi^j\right]^\dagger \left[\phi^i, \phi^j\right]\right) . \tag{17}$$

After the projection, the potential V_F remains practically the same, obviously containing only the terms which describe interactions of the surviving fields. We also have a contribution to the total scalar potential from the D-terms, that is

$$V_D = \frac{1}{2}D^2 = \frac{1}{2}D^I D_I \,, \tag{18}$$

with the D-terms having the form

$$D^I = \phi_i^\dagger T^I \phi^i \,, \tag{19}$$

where T^I are the generators in the representation of the corresponding chiral multiplets. Obviously, vanishing both F-terms and D-terms, which means $\left[\phi^i, \phi^j\right] = 0$, we obtain the minimum of the full scalar potential, i.e. all scalar fields vanish in the vacuum and therefore no spontaneous supersymmetry breaking takes place. However, interesting vacua are achieved by inserting soft supersymmetry breaking terms in the theory. Specifically, the scalar part of the soft supersymmetric breaking sector is

$$V_{SSB} = \frac{1}{2}\sum_i m_i^2 \phi^{i\dagger}\phi^i + \frac{1}{2}\sum_{i,j,k} h_{ijk}\phi^i\phi^j\phi^k + h.c. \,, \tag{20}$$

which obeys the orbifold symmetry. Therefore, the expression for the full scalar potential of the theory becomes now

$$V = V_F + V_D + V_{SSB} \,, \tag{21}$$

which can be equivalently written in the form

$$V = \frac{1}{4}(F^{ij})^\dagger F^{ij} + V_D \,, \tag{22}$$

for suitable parameters, having also defined

$$F^{ij} = \left[\phi^i, \phi^j\right] - i\varepsilon^{ijk}(\phi^k)^\dagger \,. \tag{23}$$

Due to the fact that the first term is always positive, in order to obtain the global minimum of the potential, the following equations must hold

$$\left[\phi^i, \phi^j\right] = i\varepsilon_{ijk}(\phi^k)^\dagger \,, \tag{24}$$

$$\left[(\phi^i)^\dagger, (\phi^j)^\dagger\right] = i\varepsilon_{ijk}\phi^k \,, \tag{25}$$

$$\phi^i(\phi^i)^\dagger = R^2 \,, \tag{26}$$

where $(\phi^i)^\dagger$ is the hermitian conjugate of the ϕ^i and $[R^2,\phi^i] = 0$. The above relations are related to the fuzzy sphere. This can be easily understood if we consider the twisted fields $\tilde{\phi}_i$, which are defined by

$$\phi^i = \Omega\tilde{\phi}^i\,, \tag{27}$$

for $\Omega \neq 1$, satisfying the relations

$$\Omega^3 = 1\,, [\Omega,\phi^i] = 0\,,\, \Omega^\dagger = \Omega^{-1}\,,\\ (\tilde{\phi}^i)^\dagger = \tilde{\phi}^i \Leftrightarrow (\phi^i)^\dagger = \Omega\phi^i. \tag{28}$$

Therefore, (24) converts to the relation of the ordinary sphere

$$\left[\tilde{\phi}^i,\tilde{\phi}^j\right] = i\varepsilon_{ijk}\tilde{\phi}^k\,, \tag{29}$$

which is generated by $\tilde{\phi}^i$ and (26) becomes

$$\tilde{\phi}^i\tilde{\phi}^i = R^2\,. \tag{30}$$

Expressions of ϕ^i which satisfy (24) have the following form

$$\phi^i = \Omega(\mathbf{1}_3 \otimes \lambda^i_{(N)})\,, \tag{31}$$

where $\lambda^i_{(N)}$ are the generators of the $SU(2)$ group in the $N-$dimensional representation. The matrix Ω is

$$\Omega = U \otimes \mathbf{1}_N\,, \qquad U = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\,, \qquad U^3 = \mathbf{1}\,. \tag{32}$$

The true meaning of the above configuration is revealed by diagonalizing the matrix Ω, that is

$$\tilde{\Omega}_3 := U^{-1}\Omega U = \mathrm{diag}(1,\omega,\omega^2)\,. \tag{33}$$

Therefore, (31) now becomes

$$\phi^i = \begin{pmatrix} \lambda^i_{(N)} & 0 & 0 \\ 0 & \omega\lambda^i_{(N)} & 0 \\ 0 & 0 & \omega^2\lambda^i_{(N)} \end{pmatrix}\,. \tag{34}$$

This form of ϕ^i indicates that there are actually three identical fuzzy spheres, which are embedded with relative angles $2\pi/3$.

The solution, (31), breaks completely the gauge symmetry $SU(N)^3$ (it could be considered as the Higgs mechanism of the SYM theory), yet there exists a class of solutions which do not break the gauge symmetry completely, namely

$$\phi^i = \Omega\left(\mathbf{1} \otimes (\lambda^i_{(N-n)} \oplus \mathbf{0}_n)\right)\,, \tag{35}$$

where $\mathbf{0}_n$ is the $n \times n$ matrix with zero entries. In this case, the gauge symmetry breaks from $SU(N)^3$ to $SU(n)^3$ with the vacuum being interpreted as $\mathbb{R} \times K_F$, with an internal fuzzy geometry, K_F, of a set of twisted fuzzy spheres (in ϕ^i coordinates). It is possible that this kind of vacua leads to a low-energy theory of high phenomenological interest, see discussion in [38].

Summing up, we should emphasize the general picture of the theoretical model. At very high-scale regime, we have an unbroken renormalizable gauge theory. After the spontaneous symmetry breaking, the resulting gauge theory is an $SU(3)^3$ GUT, accompanied by an unsurprising finite tower of massive Kaluza-Klein modes. Finally, the trinification $SU(3)^3$ GUT breaks down to MSSM in the low scale regime.

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Д. Гавриил, Г. Манолакос, Д. Зупанос

РЕАЛИСТИЧЕСКИЕ МОДЕЛИ ИЗ ТЕОРИЙ С ВЫСШИМИ ИЗМЕРЕНИЯМИ С НЕЧЕТКИМИ ДОПОЛНИТЕЛЬНЫМИ РАЗМЕРНОСТЯМИ

Мы даем краткий обзор программы косетной размерной редукции и наилучшие модели, построенные таким образом на этой основе. Затем мы представляем некоторые детали рассматриваемой программы в случае, когда дополнительные размерности являются нечеткими. В частности мы представляем четырехмерную $\mathcal{N} = 4$ суперсимметричную теорию поля Янга-Миллса на $\mathbb{Z}_3$ орбифолде, которая имитирует поведение размерно редуцированной десятимерной $\mathcal{N} = 1$ суперсиммтричной калибровочной теории на наборе нечетких сфер при промежуточных масштабах и ведет к тринификации теории большого объединения с калибровочной группой $SU(3)^3$ на более низких масштабах и которая может быть спонтанно нарушена до МССМ при низких энергиях.

Ключевые слова: *косетная размерная редукция, унификация, нечеткие сферы, проекция на орбифолд.*

Гавриил Д., доктор.
Национальный технический университет.

Zografou Campus, GR-15780 Athens, Греция.
E-mail: d.gavriil@yahoo.gr

Манолакос Г., доктор.
Национальный технический университет.
Zografou Campus, GR-15780 Athens, Греция.
E-mail: giorgismanolakos@gmail.com

Зупанос Д., доктор, профессор.
Национальный технический университет.
Zografou Campus, GR-15780 Athens, Греция.
E-mail: George.Zoupanos@cern.ch