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INVERSE PROBLEM FOR STATIC ELECTROMAGNETIC FIELD IN A DIPOLE APPROXIMATION

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The following inverse problem is discussed. A static electromagnetic field generated by a limited system of charges and currents is supposed to be known with its first derivatives at a point somewhere far from the system. This allows to reconstruct the position of the system, its net charge, and the electric and magnetic moments of the system.

Keywords: *inverse problem, electromagnetic field, dipole moment, charge.*

1 Statement of the problem

By the inverse problem of electrodynamics we mean the problem of reconstructing the charge and current densities from the known electromagnetic field they create. Obviously, the solution of the inverse problem of electrodynamics depends substantially on the presence of materials, the region in which the electromagnetic field is specified, ets. It is well known that the value of an analytic function and all its derivatives at some point allow to reconstruct the value of the function in the domain of definition by use of the Taylor series. Applied to the electrostatic inverse problem it means that if we know the value of electromagnetic field in some point and all its derivatives at this point, we can reconstruct the field in the whole space. Furthermore, the Maxwell equations allow to calculate the charge and current density functions $\rho(\boldsymbol{r})$ and $\boldsymbol{j}(\boldsymbol{r})$ respectively:

$$\rho(\boldsymbol{r}) = \frac{1}{4\pi}\mathrm{div}\boldsymbol{E}(\boldsymbol{r}), \quad \boldsymbol{j}(\boldsymbol{r}) = \frac{c}{4\pi}\mathrm{rot}\boldsymbol{H}(\boldsymbol{r}).$$

Here $\boldsymbol{E}(\boldsymbol{r})$ is the electric field and $\boldsymbol{H}(\boldsymbol{r})$ is the magnetic field.

In actual practice we are able to measure the field and only few first derivatives with an inevitable error. This means that we can reconstruct the field only in a small vicinity around the point where the field is measured. But the real problem encountered in practice is to calculate the field and charge distribution far from the point where the field is measured. In order to do it we have to know the value of field and its derivatives up to a very high order and with high accuracy. We present here another approach based on the multipole expansion of the field far from the static collection of charges and currents.

Suppose we know the value of electric $\boldsymbol{E}$ and magnetic $\boldsymbol{H}$ fields at some point far from limited system of charges and currents. We know also the first derivatives of fields $E_{ij} = \partial E_i/\partial x_j$ and $H_{ij} = \partial H_i/\partial x_j$. There is to define the charge q, the electric $\boldsymbol{d}$ and magnetic $\boldsymbol{m}$ dipole moments of the distant electromagnetic system, and the location $\boldsymbol{r}$ of the system. Let the coordinate origin be at the point where the fields are defined. We start with the well known formula for the fields of electric and magnetic dipoles [1, 2]

$$\boldsymbol{E} = -\frac{q\boldsymbol{r}}{r^3} + \frac{3(\boldsymbol{rd})\boldsymbol{r} - r^2\boldsymbol{d}}{r^5}, \tag{1}$$

$$\boldsymbol{H} = \frac{3(\boldsymbol{rm})\boldsymbol{r} - r^2\boldsymbol{m}}{r^5}. \tag{2}$$

Taking the derivatives of these expressions with respect to coordinates we find

$$E_{ij} = \frac{\partial E_i}{\partial x_j} = \delta_{ij}\left[-\frac{q}{r^3} + 3\frac{\boldsymbol{rd}}{r^5}\right] + 3\frac{d_j r_i + d_i r_j}{r^5} + 3r_i r_j\left[\frac{q}{r^5} - 5\frac{\boldsymbol{rd}}{r^7}\right], \tag{3}$$

$$H_{ij} = \frac{\partial H_i}{\partial x_j} = \delta_{ij}3\frac{\boldsymbol{rm}}{r^5} + 3\frac{m_j r_i + m_i r_j}{r^5} - 15r_i r_j\frac{\boldsymbol{rm}}{r^7}, \tag{4}$$

where δ_{ij} is the Kronecker symbol. One can see from Eqs (3) and (4) that the tensors E_{ij} and H_{ij} are symmetrical. This follows also from the Maxwell equations, namely from $\mathrm{rot}\boldsymbol{E} = 0$ and $\mathrm{rot}\boldsymbol{H} = 0$. Besides, the Maxwell equations $\mathrm{div}\boldsymbol{E} = 0$ and $\mathrm{div}\boldsymbol{H} = 0$ give

$$E_{11} + E_{22} + E_{33} = 0, \quad H_{11} + H_{22} + H_{33} = 0. \tag{5}$$

We consider expressions (1) – (4) as a set of equations in 10 scalar unknowns q, $\boldsymbol{r}$, $\boldsymbol{d}$ and $\boldsymbol{m}$. But this set consist of 16 equations with regard to the conditions (5), and symmetry of the tensors E_{ij} and H_{ij}. This allows us to formulate the inverse problem in more general sence. Namely, we can consider fields $\boldsymbol{E}$ and $\boldsymbol{H}$ as being originated by different sources. Let us denote by $\boldsymbol{r}_q$ the radius-vector of a charge collection with a total charge q and an electric dipole moment

$\boldsymbol{d}$, and by $\boldsymbol{r}_m$ the radius-vector of a system of currents with a magnetic dipole moment $\boldsymbol{m}$. Then the set of equations (1) – (4) splits into two independent sets for $\boldsymbol{r}_q$ and $\boldsymbol{r}_m$. In solving these equations we can obtain in particular $\boldsymbol{r}_q = \boldsymbol{r}_m$, which means that the electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{H}$ are generated by the same electromagnetic system.

It should be pointed out, that in calculation of fields far from an electromagnetic system we can neglect the higher terms of multipole expansion. The accuracy of such representation depends on the ratio between the sizes of the system and the distance between the system and observer. In solving the inverse problem we find only the distance $|\boldsymbol{r}|$, but not the sizes of system. In order to estimate the accuracy of the received solution one have to calculate the sizes of the system by some independent method. For example one can solve the inverse problem for a few different positions of the observer and then one can estimate the sizes of the system. One can also investigate the field in the vicinity of the observer in order to find out whether this field is of dipole nature. These problems are not considered in this paper. In order to exclude such questions we consider further a point-like charge with an electric dipole moment and a point-like magnetic moment. In the next two sections we solve the inverse problem first for the equations (1) and (3) and then for equations (2) and (4).

2 Inverse problem for a charge and electric dipole

Let us solve the equations (1) and (3) with respect to the unknown q, $\boldsymbol{d}$ and $\boldsymbol{r}$[1]. We choose the coordinate system as follows: the coordinate origin is placed at the point where the field is specified, the x axis is directed along the vector $\boldsymbol{E}$ and the y axis is aligned with the principal normal to the electric field line. The vector of the principal normal $\boldsymbol{n}$ is defined by the equality [3]

$$\boldsymbol{n} = \frac{1}{kE}\frac{\partial \boldsymbol{E}}{\partial s},$$

where k is the curvature of the field line and ∂s is the displacement along the field line. Taking derivative from $\boldsymbol{E}$ we find

$$\boldsymbol{n} = \frac{1}{kE^4}[\boldsymbol{E}[\boldsymbol{D}\boldsymbol{E}]],$$

where $\boldsymbol{D} = (\boldsymbol{E}\boldsymbol{\nabla})\boldsymbol{E}$ with the projections $D_i = E_k E_{ik}$ (a summation over repeated indices is implied). The projections of the vector $\boldsymbol{n}$ on the axes of an arbitrary orthogonal coordinate system are expressed in terms of components of the tensor E_{ij} as follows:

$$n_i = \frac{1}{kE^4} E_k E_j (E_j E_{ik} - E_i E_{kj}).$$

If the x axis is directed in the sense of the vector $\boldsymbol{E}$, then

$$\boldsymbol{n} = \frac{1}{kE}\{0,\ E_{12},\ E_{13}\}. \tag{6}$$

With the y axis directed in the sense of the vector $\boldsymbol{n}$ we have $E_{13} = 0$ and $E_{12} > 0$. It can be seen from Eq. (1) that the vectors $\boldsymbol{E}$, $\boldsymbol{d}$ and $\boldsymbol{r}$ lie in the same plane; therefore, the problem is reduced to a two-dimensional one. In the x, y, z coordinate system we have $\boldsymbol{E} = (E, 0, 0)$, $\boldsymbol{d} = (d_x, d_y, 0)$, and $E_{23} = 0$ as a consequence of the axial symmetry of the field. Thus, five functions of the coordinates E_x, E_y, E_{11}, E_{12} and E_{33} are known. They can be used to find five unknowns q, d_x, d_y, r_1 and r_2. Eqs (1) and (3) in the (x, y) plane take the form

$$E = -\frac{qx}{r^3} + \frac{3(\boldsymbol{r}\boldsymbol{d})x - r^2 d_x}{r^5}, \tag{7}$$

$$0 = -\frac{qy}{r^3} + \frac{3(\boldsymbol{r}\boldsymbol{d})y - r^2 d_y}{r^5}, \tag{8}$$

$$E_{11} = \frac{q}{r^3} - 3\frac{(\boldsymbol{r}\boldsymbol{d})}{r^5} - 6\frac{d_x x}{r^5} - 3x^2\left[\frac{q}{r^5} - 5\frac{(\boldsymbol{r}\boldsymbol{d})}{r^7}\right], \tag{9}$$

$$E_{12} = -3\frac{d_x y + d_y x}{r^5} - 3xy\left[\frac{q}{r^5} - 5\frac{(\boldsymbol{r}\boldsymbol{d})}{r^7}\right], \tag{10}$$

$$E_{33} = \frac{q}{r^3} - 3\frac{(\boldsymbol{r}\boldsymbol{d})}{r^5}. \tag{11}$$

The charge q and the components of the dipole moment can be easily expressed from Eqs (7), (8) and (11):

$$d_x = -r^3(E + E_{33}x), \tag{12}$$

$$d_y = -r^3 y E_{33}, \tag{13}$$

$$q = -2r^3 E_{33} - 3xrE. \tag{14}$$

Substituting these formulas into Eqs (9) and (10), we find

$$E_{11} = E_{33} + 6\frac{xy^2 E}{r^4} - 3\frac{x^2 E_{33}}{r^2},$$

$$E_{12} = -3\frac{xyE_{33}}{r^2} + 3\frac{yE}{r^2} - 6\frac{yx^2 E}{r^4}. \tag{15}$$

Taking into account that $r = \sqrt{x^2 + y^2}$, we get two equations depending only on x and y. Thus, the problem is reduced to the solution of the last system of equations (15). By simple algebraic manipulations this system can be reduced to the following form ($r \neq 0$):

$$x^2(2E_{33} + E_{11}) + y^2(E_{33} - E_{11}) + 2yxE_{12} = 0,$$

$$-xy(E_{33} + 2E_{11}) - y^2 E_{12} + x^2 E_{12} + 3yE = 0. \tag{16}$$

Let us introduce the designations

$$F_1 = E_{22} - E_{33},\ \ F_2 = E_{33} - E_{11},\ \ F_3 = E_{11} - E_{22},$$

$$S = \sqrt{E_{12}^2 + F_1 F_2}.$$

[1] We omit the subscripts q and m having in mind that in the section 2 we are dealing only with $\boldsymbol{r}_q$ and in section 3 only with $\boldsymbol{r}_m$.

Then the solution of Eq. (16) can be written in the form

$$x = \frac{3E}{4E_{12}^2 + F_3^2}\left[F_3 \pm \frac{E_{12}(F_1 - F_2)}{S}\right], \quad (17)$$

$$y = \frac{3E}{4E_{12}^2 + F_3^2}\left[2E_{12} \pm \frac{2E_{12}^2 - F_1F_3}{S}\right]. \quad (18)$$

In order to calculate the dipole moment, we find the squared radius-vector r

$$r^2 = \frac{9E^2}{S^2(4E_{12}^2 + F_3^2)}\left[2E_{12}^2 - F_1F_3 \pm 2SE_{12}\right]. \quad (19)$$

Substituting Eqs (17), (18) and (19) into Eq. (12), we find the dipole moment

$$\begin{aligned} d_x &= \mp E^{10}(E_{12}(F_1^2 + (-E_{12} \pm S)^2) \mp 2F_1^2S) \\ &\times \frac{27F_1^3}{(S)^4(F_1^2 + (-E_{12} \pm S)^2)^{5/2}}, \quad (20) \\ d_y &= \pm\frac{81E^4F_1^5E_{33}}{(S)^4(F_1^2 + (-E_{12} \pm S)^2)^{5/2}}. \quad (21) \end{aligned}$$

The sign $\pm$ in the formulas indicates the existence of two solutions of the initial system of equations (7-11). Physically this means that the same field with its derivatives may be created at the given point by two different sources located at different places. One can find some specific cases of this solution in Ref. [4].

3 Inverse problem for a magnetic dipole moment

In order to find the position vector $\boldsymbol{r}$ and the magnetic moment $\boldsymbol{m}$ of a particle generating the field $\boldsymbol{H}$ we solve equations (2) and (4). Now we align the axis x along the vector $\boldsymbol{H}$ and the axis y along the principal normal to the magnetic field line. Repeating the reasoning of the previous section we get

$$\begin{aligned} H &= \frac{3(\boldsymbol{rm})x - r^2m_x}{r^5}, \quad (22) \\ 0 &= \frac{3(\boldsymbol{rm})y - r^2m_y}{r^5}, \quad (23) \\ H_{11} &= -3\frac{(\boldsymbol{rm})}{r^5} - 6\frac{m_xx}{r^5} + 15x^2\frac{(\boldsymbol{rm})}{r^7}, \quad (24) \\ H_{12} &= -3\frac{m_xy + m_yx}{r^5} + 15xy\frac{(\boldsymbol{rm})}{r^7}, \quad (25) \\ H_{33} &= -3\frac{(\boldsymbol{rm})}{r^5}. \quad (26) \end{aligned}$$

Thus, we have 5 equations for 4 unknowns. Hence, the system of equations (22) – (26) is a overdetermined one. It means that the components of $\boldsymbol{H}$ and H_{ij} are not independent. The fact that the field $\boldsymbol{H}$ is produced by a magnetic moment places a constraint on $\boldsymbol{H}$ and H_{ij}. The corresponding equation will be found latter (see Eq. (39)). Substituting $(\boldsymbol{rm})$ from Eq. (26) into Eqs (22) – (25), we find

$$\begin{aligned} H &= -H_{33}x - \frac{m_x}{r^3}, \quad (27) \\ 0 &= -H_{33}y - \frac{m_y}{r^3}, \quad (28) \\ H_{11} &= H_{33} - 6\frac{m_xx}{r^5} - 5x^2\frac{H_{33}}{r^2}, \quad (29) \\ H_{12} &= -3\frac{m_xy + m_yx}{r^5} - 5xy\frac{H_{33}}{r^2}. \quad (30) \end{aligned}$$

Eliminating m_x and m_y we get

$$\begin{aligned} H_{11} &= H_{33} + x^2\frac{H_{33}}{r^2} - 6x\frac{H}{r^2}, \quad (31) \\ H_{12} &= 3\frac{H}{r^2}y + xy\frac{H_{33}}{r^2}, \quad (32) \end{aligned}$$

and for the square of the distance

$$r^2 = -\frac{3Hx}{2H_{33}}. \quad (33)$$

We suppose here that $H_{33} \neq 0$. The case $H_{33} = 0$ will be considered latter. Using Eqs (31) – (32) one can find

$$x = -\frac{3H(3H_{33} + H_{11})}{2H_{33}^2}, \quad (34)$$

$$y = -\frac{3HH_{12}(3H_{33} + H_{11})}{2H_{33}^2(H_{11} + 5H_{33})}. \quad (35)$$

Now we express r from Eq. (33)

$$|r| = \frac{3|H|}{2|H_{33}|}\sqrt{\frac{3H_{33} + H_{11}}{H_{33}}}.$$

And hence

$$m_x = \frac{1}{3}\left(\frac{3H}{2H_{33}}\right)^4(7H_{33} + 3H_{11})\left(\frac{3H_{33} + H_{11}}{H_{33}}\right)^{3/2}, \quad (36)$$

$$m_y = H_{12}\left(\frac{3H}{2H_{33}}\right)^4\frac{3H_{33} + H_{11}}{H_{11} + 5H_{33}}\left(\frac{3H_{33} + H_{11}}{H_{33}}\right)^{3/2}. \quad (37)$$

Let us consider the specific case $H_{33} = 0$. It follows from Eq. (26) that $(\boldsymbol{rm}) = 0$, which gives immediately $x = 0$. Eqs (22) – (25) give the whole solution in this case

$$x = 0, \quad y = 3\frac{H}{H_{12}}, \quad m_x = -27\frac{H^4}{H_{12}^3}, \quad m_y = 0. \quad (38)$$

Thus, Eqs (34) – (37) and in particular case Eq. (38) give the solution of the problem.

It was mentioned above that the system of Eqs (22) – (26) is a overdetermined one. Let us find relation between $\boldsymbol{H}$ and H_{ij}. On the one hand it follows from Eq. (33) that

$$r^2 = \frac{9H^2(3H_{33} + H_{11})}{4H_{33}^3}, \quad (H_{33} \neq 0).$$

On the other hand relations (34) and (35) give

$$r^2 = \frac{9H^2(3H_{33}+H_{11})^2}{4H_{33}^4}\left(1+\frac{H_{12}^2}{H_{33}^4}\frac{1}{(H_{11}+5H_{33})^2}\right).$$

This gives the desired equation

$$\frac{H_{33}}{3H_{33}+H_{11}} = 1+\frac{H_{12}^2}{(H_{11}+5H_{33})^2}. \qquad (39)$$

In case $H_{33}=0$ we have from Eqs (22) – (26) $H_{11}=0$.

If the condition (39) is not fulfilled, the field $\boldsymbol{H}$ is not produced by a magnetic dipole.

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ОБРАТНАЯ ЗАДАЧА ДЛЯ СТАЦИОНАРНОГО ЭЛЕКТРОМАГНИТНОГО ПОЛЯ В ДИПОЛЬНОМ ПРИБЛИЖЕНИИ

Обсуждается следующая обратная задача. Предполагается известным электромагнитное поле и его первые производные по координатам для ограниченной системы зарядов и токов в некоторой точке вдали от этой системы. Это позволяет локализовать систему и определить ее полный заряд, электрический и магнитный дипольные моменты.

Ключевые слова: *обратная задача, электромагнитное поле, дипольный момент, заряд.*

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