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COMPLEX COORDINATE TRANSFORMATIONS TO GET EXACT AXIALLY SYMMETRIC SOLUTIONS IN $F(R)$-GRAVITY

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We present a strategy to get axially symmetric solutions in $f(R)$ gravity by starting from spherically symmetric space-times. To do so, we assume the validity of a complex coordinate transformation, which acts on the spherically symmetric metric and permits one to infer the corresponding $f(R)$ modification. The consequences of this recipe are here described, giving particular emphasis to define a class of compatible axially symmetric solutions, which fairly well describe the motion in cylindrical geometries in the field of $f(R)$, in two different classes of coordinates. We demonstrate that our approach is general and may be applied for several cases of interest. We also show that our treatment is compatible with the standard approach of general relativity, evaluating the motion of a freely falling particle in the context of our metric.

Keywords: *complex coordinate transformations, axially symmetric solutions, $f(R)$-gravity.*

1 Introduction

Modified theories of gravity have been introduced to address several inconsistencies that general relativity seems to be unable to explain [1]. In particular, the issue of understanding in which manners exact solutions may be modified, involving alternative theories of gravity has become a highly relevant topic [2]. Moreover, the need of understanding whether those theories may reproduce the general relativity results is essential to check their viabilities. Among all the possibilities, we circumscribe our attention to $f(R)$-gravity, where $f(R)$ represents an analytic function the Ricci scalar R. Those models satisfy the equivalence principle and candidate as a serious alternative to standard Einstein's gravity, especially for describing both dark matter and dark energy effects at different cosmic scales [3]. The great disadvantage of those paradigms is that $f(R)$ is not known *a priori*. Hence, reconstructing $f(R)$ functions by consistently matching their shapes with present-time data has become essential for their determinations. Here, we investigate some classes of exact solutions. Their importance lies on the fact that in order to describe astrophysical compact objects, e.g. black holes or active galactic nuclei and so forth, those solutions may represent a key point. Rephrasing it differently, alternative $f(R)$ theories should be consistent with general relativity outcomes, reproducing standard exact solutions, i.e. Schwarzschild spherically symmetric solution or Kerr axially symmetric solution, etc. In addition, getting new suitable solutions may be of great physical interest, since it could represent a technique for revealing $f(R)$ effects at high energy astrophysical regimes, fixing additional bounds on the $f(R)$ determination. Thus, strategies to find out either exact or approximate solutions in modified gravities become profoundly significative for a whole understanding of $f(R)$ gravities. To better clarify this fact, we may start from recent developments spanning from spherically symmetric solutions of $f(R)$-gravity in vacuum [4] to static spherically symmetric formulations in presence of perfect matter fluid in metric formalism picture [5, 6]. Keeping in mind those results, we extend the standard formalism by proposing a more complete treatment to find out axially symmetric solutions, by means of a *complex coordinate transformation* acting on the spherically symmetric metric [7, 8]. In particular, interior solutions have not so far obtained to characterize a whole exact solution. The problem lies on the loss of a symmetry degrees which makes the corresponding derivation of any solution highly complicated. Among all techniques, we employ the Newman and Janis treatment, which seems to alleviate the problem of losing symmetry degrees. They argued that one may obtain an axially symmetric solution, i.e. a Kerr-like metric, by considering a *complex transformation* on the spherical solution [9, 10]. In [11] a self consistent and rigorous proof that the Kerr metric can be effectively determined from a complex transformation on the Schwarzschild solution, has been given. Here, we extend such a formalism, showing that the complex transformation may be framed in the context of $f(R)$ gravities [12]. The paper is structured as follows. In Sec. II, we describe the method and we highlight its fundamental properties. To do so, we consider the general treatment and we specialize it to the case of pure spherically symmetric solutions. We therefore obtain the corresponding modifications to the standard Kerr metric in the context of $f(R)$ gravity and we describe some dynamical properties of this solution, by means of circular orbits in the framework of the Hamiltonian formalism. We therefore demonstrate that our strategy is general and may be extended to the case of fourth order gravities without stability problems. In Sec. III, we summarize our results and we propose possible perspectives of our method.

2 From spherical symmetry to axially symmetric solutions in $f(R)$ gravity

In the framework of $f(R)$ gravity, the action takes the simple form $S = \int d^4x\sqrt{-g}\Big[f(R) + \mathcal{X}\mathcal{L}_m\Big]$. By varying it, in terms of the metric $g_{\mu\nu}$, one argues the corresponding field equations:

$$\begin{cases} f_{\mu\nu} - f'(R)_{;\mu\nu} + g_{\mu\nu}\Box f'(R) \,=\, \mathcal{X}T_{\mu\nu}\,, \\ f_{\mu\nu} \equiv f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu}\,, \\ 3\Box f'(R) + f'(R)R - 2f(R) \,=\, \mathcal{X}T\,, \end{cases} \qquad (1)$$

where $T_{\mu\nu}$ represents the standard energy-momentum tensor for dust-like matter, which can be expressed in the form: $T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}$. The constant $\mathcal{X}$ contains the gravitational constant G, since $\mathcal{X} = \frac{8\pi G}{c^4}$, while g is the metric determinant.

Our formalism involves the use of spherically symmetric space-time as starting point. In fact, we set up our treatment by assuming the most general spherically symmetric space-time below:

$$ds^2 \,=\, g_{tt}(t,r)dt^2 - g_{rr}(t,r)dr^2 - r^2 d\Omega\,, \qquad (2)$$

in which $d\Omega$ represents the solid angle. The basic demands consists in employing on it a transformation that maps Eq. (2), providing that the off-diagonal terms vanish. Hence, the spherically symmetric space-time may be obtained by assuming that Eq. (2) satisfies particular cosmic symmetries. Here, we consider the Noether symmetries and so, after several calculations, we can write down the simplest spherically symmetric space-time as:

$$ds^2 = (\alpha + \beta r)dt^2 - \frac{1}{2}\frac{\beta r}{\alpha + \beta r}dr^2 - r^2 d\Omega\,, \qquad (3)$$

where we assumed α as a combination of auxiliary constants, e.g. Σ_0 and k and $\beta = k_1$ [12].

Here, we demonstrate how it is possible to get an axially symmetric solution adopting the Newman-Janis procedure, extending their treatment in the context of $f(R)$ gravities and going beyond the standard usage of using the Newman-Janis procedure in general relativity only. To this end, as we already stressed before, we employ the existence of Noether symmetries which make the $f(R)$ model consistent with the corresponding field equations. For our purposes, let us recast the spherically symmetric metric as $ds^2 = e^{2\phi(r)}dt^2 - e^{2\lambda(r)}dr^2 - r^2 d\Omega$, with $g_{tt}(t,r) = e^{2\phi(r)}$ and $g_{rr}(t,r) = e^{2\lambda(r)}$. Hereafter, our convention is to refer to time-like components as tt or 00, whereas space-like as rr or ii, with i running from $i = 0$ to $i = 3$.

Considering the suitable Eddington–Finkelstein coordinates, i.e. (u, r, θ, ϕ), which represent a viable choice for our coordinate representation, after simple algebra, we definitively get $ds^2 = e^{2\phi(r)}du^2 \pm 2e^{\lambda(r)+\phi(r)}dudr - r^2 d\Omega$. Thus, the matrix associated to the metric is rewritable in terms of a null tetrad as:

$$g^{\mu\nu} = l^\mu n^\nu + l^\nu n^\mu - m^\mu \bar{m}^\nu - m^\nu \bar{m}^\mu\,, \qquad (4)$$

where l^μ, n^μ, m^μ and $\bar{m}^\mu$ should satisfy

$$l_\mu l^\mu \,=\, m_\mu m^\mu \,=\, n_\mu n^\mu = 0\,, \qquad (5)$$

$$l_\mu n^\mu \,=\, -m_\mu \bar{m}^\mu = 1\,, \qquad (6)$$

$$l_\mu m^\mu \,= n_\mu m^\mu = 0\,, \qquad (7)$$

where we assumed the bars as indication of the complex conjugation.

In our case, a generic space-time event becomes

$$x^\mu \to \tilde{x}^\mu = x^\mu + iy^\mu(x^\sigma)\,, \qquad (8)$$

in which we notice that $y^\mu(x^\sigma)$ are functions of the real coordinates x^σ. Analogously, the null tetrad vectors $Z_a^\mu = (l^\mu, n^\mu, m^\mu, \bar{m}^\mu)$, with $a = 1,2,3,4$, should satisfy

$$Z_a^\mu \to \tilde{Z}_a^\mu(\tilde{x}^\sigma, \bar{\tilde{x}}^\sigma) \,=\, Z_a^\rho \frac{\partial \tilde{x}^\mu}{\partial x^\rho}\,. \qquad (9)$$

All this procedure provides a net effect which consists in generating a new metric. The component of such a space-time are real and depend upon complex variables. We have:

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu}\,:\, \tilde{\mathbf{x}} \times \tilde{\mathbf{x}} \mapsto \mathbb{R}\,, \qquad (10)$$

where we consider:

$$\tilde{Z}_a^\mu(\tilde{x}^\sigma, \bar{\tilde{x}}^\sigma)|_{\mathbf{x}=\tilde{\mathbf{x}}} = Z_a^\mu(x^\sigma)\,. \qquad (11)$$

From the transformed null tetrad vectors, a new metric is therefore obtained. So, assuming the covariant form, we can list the corresponding metric components as:

$$
\begin{aligned}
g_{00} =& e^{2\phi(\tilde{r},\theta)}\,,\\
g_{01} =& e^{\lambda(\tilde{r},\theta)+\phi(\tilde{r},\theta)}\,,\\
g_{03} =& a e^{\phi(\tilde{r},\theta)}[e^{\lambda(\tilde{r},\theta)} - e^{\phi(\tilde{r},\theta)}]\sin^2\theta\,,\\
g_{13} =& - a e^{\phi(\tilde{r},\theta)+\lambda(\tilde{r},\theta)}\sin^2\theta\,,\\
g_{22} =& -\Sigma^2\,,\\
g_{33} =& -[\Sigma^2 + a^2\sin^2\theta e^{\phi(\tilde{r},\theta)}(2e^{\lambda(\tilde{r},\theta)} - e^{\phi(\tilde{r},\theta)})]\sin^2\theta.
\end{aligned}
$$

Where we assumed that all the other components, i.e. the components that we did not report above, are zero. This procedure is circumscribed to the use of the particular choice of coordinates. However, one can also perform the Newman-Janis algorithm on any static spherically symmetric solutions, by means of the more practically Boyer-Lindquist coordinates. So, evaluating the same steps performed above and the analogous strategy to get the tetrad null vectors in the case of axially symmetric space-time, we simply obtain:

$$
\begin{aligned}
g_{00} =& \frac{r(\alpha+\beta r) + a^2\beta\cos^2\theta}{\Sigma}\,,\\
g_{03} =& \frac{a(-2\alpha r - 2\beta\Sigma^2 + \sqrt{2\beta}\Sigma^{3/2})\sin^2\theta}{2\Sigma}\,,\\
g_{11} =& -\frac{\beta\Sigma^2}{2\alpha r + \beta(a^2 + r^2 + \Sigma^2)}\,,\\
g_{22} =& -\Sigma^2\,,\\
g_{33} =& -\left[\Sigma^2 - \frac{a^2(\alpha r + \beta\Sigma^2 - \sqrt{2\beta}\Sigma^{3/2})\sin^2\theta}{\Sigma}\right]\sin^2\theta.
\end{aligned}
$$

Again all components, which do not appear above, are zero.

As in standard general relativity, our treatment should be compatible with the motion of a freely falling particle. Hence, we can treat a physical example which accounts for a freely falling particle moving in our so-obtained metric. To do so, we make extensive use of the Hamiltonian formalism, which has the advantage not to show any sign ambiguity which may come from turning points in the orbits [13]. The reduced Hamiltonian, linearly reported in terms of momenta, is:

$$
H = \left[\frac{p_i g^{0i}}{g^{00}} + \left[\left(\frac{p_i g^{0i}}{g^{00}}\right)^2 - \frac{m^2 + p_i p_j g^{ij}}{g^{00}}\right]^{1/2}\right], \quad (12)
$$

providing $H = -p_0$ and even satisfying the following motion equations:

$$
\frac{dx^i}{dt} = \frac{\partial H}{\partial p_i}\,, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x^i}\,, \quad (13)
$$

which permit to numerically obtain the requested orbits. In particular, in the equatorial plane, which corresponds to the case $\theta = \frac{\pi}{2}$, $\dot{\theta} = 0$, we conventionally employ $\alpha = 1$ and $\beta = 2$, without losing generality and we consider the dependence on ϕ and on the conjugate momentum p_ϕ, which represents an integral of motion. As a consequence, we find out that the coupled equations for $\{r, \theta, \phi, p_r, p_\theta\}$ may be numerically integrated, giving compatible trajectories with respect to the ones inferred from the standard Kerr space-time. To better clarify this statement, we explicitly report below the geodesic equations:

$$
\frac{dx^\mu}{d\lambda} = \frac{\partial\mathcal{H}}{\partial p_\mu} = g^{\mu\nu}p_\nu = p^\mu\,, \quad (14)
$$

$$
\frac{dp_\mu}{d\lambda} = -\frac{\partial\mathcal{H}}{\partial x\mu} = -\frac{1}{2}\frac{\partial g^{\alpha\beta}}{\partial x^\mu}p_\alpha p_\beta = g^{\gamma\beta}\Gamma^\alpha_{\mu\gamma}p_\alpha p_\beta\,, \quad (15)
$$

and also the corresponding reduced Hamiltonian:

$$
\begin{aligned}
H =& \frac{2ap_\phi\left(-2r^3 + r^2 - 1\right)}{a^2\left(-2(r-1)r^2 - 1\right) + r^5}\\
&+ \{\mathcal{A}(a, p_\phi, r)\mathcal{B}(a, p_\phi, r)\\
&\times\ (\mathcal{C}(a, p_\phi, r) - \mathcal{D}(a, p_\phi, r) - p_r + 1)\}^{\frac{1}{2}}\,,
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{A}(a, p_\phi, r) =& 4a^2p_\phi^2\left(-2r^3 + r^2 - 1\right)^2\\
&-a^2\left(-2(r-1)r^2 - 1\right) - r^5\,,
\end{aligned}
$$

$$
\mathcal{B} = \left(a^2\left(r^2(r(2r-3)(2r+1)+6) - 2\right) + (2r+1)r^4\right)\,,
$$

and

$$
\begin{aligned}
&\mathcal{C}(a, p_\phi, r) =\\
&= \frac{-p_\phi(2r+1)}{a^2\left(r^2(r(2r-3)(2r+1)+6) - 2\right) + (2r+1)r^4}\,,
\end{aligned}
$$

$$
\mathcal{D}(a, p_\phi, r) = \frac{p_r\left(a^2 + r^2 + r\right) + p_\theta}{r^4}\,.
$$

Soon, it is evident that H is independent from ϕ and, as already above stressed, the conjugate momentum p_ϕ is an integral of motion. Finally, the numerical results may be found in [12].

3 Final outlooks and perspectives

In this paper, we considered the framework of $f(R)$ gravity to describe a technique able to get axially symmetric solutions from spherical ones. This treatment has been extensively described by Newman-Janis in a precise algorithm, which takes into account complex transformations. In particular, assuming a spherically symmetric expression for the space-time, we demonstrated that it is possible to extend the complex transformations in the context of $f(R)$ gravity. To do so, we evaluated the null tetrad associated to this method in two different classes of coordinates and we found out the corresponding axially symmetric metrics. In order to understand if the thus obtained space-time works well in the field of particle motion, we considered a freely falling particle and we showed that its motion is perfectly compatible with the expected standard Kerr metric, which corresponds to the simplest axially symmetric solution in general relativity. Further investigations will be carried forward in order to describe different symmetries by means of the Newman-Janis strategy. In particular, measuring possible corrections due to $f(R)$ around compact objects, e.g. evaluating possible discrepancies from the standard cases of accretion disks, one would constrain the $f(R)$ functions at astrophysical regimes. This would open new challenges for the problem of $f(R)$ reconstructions.

References

[1] Capozziello S. and De Laurentis M. 2011 *Phys. Rept.* **509** 167.

[2] Bamba K., Capozziello S., Nojiri S. and Odintsov S. D. 2012 *Astrophys. Space Sci.* **342** 155-228.

[3] Capozziello S., De Laurentis M., Luongo O. and Ruggeri A. C. 2013 *Galaxies* **1** 216.

[4] Multamaki T. and Vilja I. 2006 *Phys. Rev. D* **74** 064022.

[5] Multamaki T. and Vilja I. 2007 *Phys. Rev. D* **76** 064021.

[6] Kainulainen K., Piilonen J., Reijonen V., and Sunhede D. 2007 *Phys. Rev. D* **76** 024020.

[7] McManus D. 1991 *Class. Quant. Grav.* **8** 863.

[8] Kransinski A. 1978 *Ann. Phys.* **112** 22.

[9] Newman E. T. and Janis A. I. 1965 *J. Math. Phys.* **6** 915.

[10] Newman E. T., Couch E., Chinnapared K., Exton A., Prakash A. and Torrence R. 1965 *J. Math. Phys.* **6** 918.

[11] Schiffer M. M., Adler R. J., Mark J. and Sheffield C. 1973 *J. Math. Phys.* **14** 52.

[12] Capozziello S., De laurentis M. and Stabile A. 2010 *Class. Quant. Grav.* **27** 165008.

[13] Berti E. 2014 ArXiv[gr-qc]:1410.4481.

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КОМПЛЕКСНЫЕ ПРЕОБРАЗОВАНИЯ КООРДИНАТ ДЛЯ ПОЛУЧЕНИЯ ТОЧНЫХ АКСИАЛЬНО-СИММЕТРИЧНЫХ РЕШЕНИЙ В $f(R)$-ГРАВИТАЦИИ

Мы описываем стратегию получения точных аксиально-симметричных решений в $f(r)$-гравитации начиная со сферически симметричного пространства-времени. Для этого мы предполагаем справедливость комплексных преобразований координат, действующих в сферически-симметричной метрике и допускающих введение соответствующей $f(R)$ модификации. Описаны следствия такого подхода, в частности, подчеркивается возможность получения класса совместимых аксиально-симметрических решений, которые довольно хорошо описывают движение в поле $f(R)$ цилиндрической геометрии в двух различных классах координат. Мы показываем, что наш подход является общим и применим в различных случаях. Мы также показываем, что наш метод совместим со стандартным подходом общей теории относительности при рассмотрении свободно падающей частицы в контексте нашей метрики.

Ключевые слова: *комплексные преобразования координат, аксиально симметричные решения, $f(R)$-гравитация.*

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